# Chapter 8: Maclaurin's Series

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## Outline





## Introduction

- Using Maclaurin's series, mixed functions containing, say, algebraic, trigonometric and exponential functions, may be expressed solely as algebraic functions, and differentiation and integration can often be more readily performed.
- Maclaurin's theorem or Maclaurin's series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$



## Conditions of Maclaurin's Series

At $x = 0$ the following 3 conditions are met:

- $f(0) \neq \infty$
- $f'(0), f''(0), f'''(0), \ldots \neq \infty$
- The resultant Maclaurin's series must be convergent.

## Worked Problems on Maclaurin's Series

- **Problem 4.** Expand ln(1 + x) to five terms.
  [$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots$]

- **Problem 5.** Expand ln(1 – x) to five terms.
  [$\ln(1 - x) = -x - x^2/2 - x^3/3 - x^4/4 - x^5/5 - \ldots$]

- **Problem 6.** Determine the power series for ln((1 + x)/(1 – x)).
  [$\ln((1 + x)/(1 - x)) = 2(x + x^3/3 + x^5/5 + \ldots)$]



## Worked Problems on Maclaurin's Series

- **Problem 1.** Determine the first four terms of the power series for cos x.
  [$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$]

- **Problem 10.** Produce a power series for $\cos^2 2x$ as far as the term in $x^6$.
  (Hint: $\cos 2x = 2\cos^2 x - 1$)
  [$\cos^2 2x = 1 - 4x^2 + 16x^4/3 - 128x^6/45 + \ldots$]



## Worked Problems on Maclaurin's Series

- **Problem 3.** Determine the power series for tan x as far as the term in $x^3$.
  [$\tan x = x + x^3/3$]



## Exercise 36

- **Exercise 1.** Determine the first four terms of the power series for sin 2x using Maclaurin's series.
  [$\sin 2x = 2x - 4x^3/3 + 4x^5/15 - 8x^7/315 + \ldots$]

- **Exercise 6.** Develop, as far as the term in $x^4$, the power series for sec 2x.
  [$1 + 2x^2 + 10x^4/3$]

## Numerical Integration Using Maclaurin's Series

- A method of finding the approximate value of a definite integral is to express the function as a power series using Maclaurin's series, and then integrating each algebraic term in turn.
- **Problem 11.** Evaluate $\int_{0.1}^{0.4} 2e^{\sin\theta}\,d\theta$, correct to 3 significant figures.
  [0.771]
- **Problem 12.** Evaluate $\int_{0}^{1} \frac{\sin\theta}{\theta}\,d\theta$ using Maclaurin's series, correct to 3 significant figures.
  [0.946]



## Exercise 37

- **Exercise 2.** Use Maclaurin's theorem to expand $\cos 2x$ and hence evaluate, correct to 2 decimal places, $\int_{0}^{1} \frac{\cos 2x}{x^{1/3}}\,dx$.
  [0.88]
- **Exercise 4.** Use Maclaurin's theorem to expand $\sqrt{x}\ln(x+1)$ as a power series. Hence evaluate, correct to 3 decimal places, $\int_{0}^{0.5} \sqrt{x}\ln(x+1)\,dx$.
  [0.061]



## Limiting Values (1/2)

- It is sometimes necessary to find limits of the form $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}$, where $f(a) = 0$ and $g(a) = 0$.
- For certain limits, a knowledge of series can sometimes help. (Hint: **Problem 3**)

$$\lim_{x\to 0}\left\{\frac{\tan x - x}{x^3}\right\} \equiv \lim_{x\to 0}\left\{\frac{x + x^3/3 + \ldots - x}{x^3}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{x^3/3 + \ldots}{x^3}\right\} = \lim_{x\to 0}\left\{\frac{1}{3}\right\} = \frac{1}{3}$$



## Limiting Values (2/2)

- However, a knowledge of series does not help with examples such as $\lim_{x\to 1}\left\{\frac{x^2 + 3x - 4}{x^2 - 7x + 6}\right\}$.
- L'Hopital's rule will enable us to determine such limits when the differential coefficients of the numerator and denominator can be found.

# L'Hopital's Rule

- L'Hopital's rule states:

$$\lim_{x \to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x \to a}\left\{\frac{f'(x)}{g'(x)}\right\} \text{ provided } g'(a) \neq 0.$$

- It can happen that $\lim_{x \to a}\left\{\frac{f'(x)}{g'(x)}\right\}$ is still $\frac{0}{0}$; if so, the numerator and denominator are differentiated again (and again) until a non-zero value is obtained for the denominator.



# Problems & Exercise 38

- **Problem 14.** Determine $\lim_{x \to 1}\left\{\frac{x^2 + 3x - 4}{x^2 - 7x + 6}\right\}$ [-1]
- **Problem 15.** Determine $\lim_{x \to 0}\left\{\frac{\sin x - x}{x^2}\right\}$ [0]
- **Exercise 5.** Determine $\lim_{\theta \to 0}\left\{\frac{\sin\theta - \theta\cos\theta}{\theta^3}\right\}$ [1/3]
- **Exercise 7.** Determine $\lim_{x \to 0}\left\{\frac{\sinh x - \sin x}{x^3}\right\}$ [1/3]