# Chapter 52: Power Series Methods of Solving Ordinary Differential Equations

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## Outline

- Introduction
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- Power Series Solution by The Leibniz-Maclaurin Method
- Power Series Solution by the Frobenius Method
- Bessel's Equation and Bessel's Functions
- Legendre's Equation and Legendre Polynomials

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## Introduction

- Second order ordinary differential equations that cannot be solved by analytical methods (Chapter 50 and 51), i.e., those involving variable coefficients, can often be solved in the form of an infinite series of powers of the variable.
- To better understand this chapter, it is necessary to be able to:
  - Differentiate standard functions (Chapter 27 and 32),
  - Appreciate the binomial theorem (Chapter 7), and
  - Use Maclaurin's theorem (Chapter 8)

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## Higher Order Differential Coefficients as Series (1/4)

- We abbreviate $\frac{\mathrm{d}y}{\mathrm{d}x}$ as $y'$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ as $y''$,…and $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ as $y^{(n)}$.
- If $y = \mathrm{e}^{ax}$,
  - $y^{(n)} = a^n \mathrm{e}^{ax}$
  - If $y = 3\mathrm{e}^{2x}$, then $y^{(7)} = 384\mathrm{e}^{2x}$
- If $y = \sin ax$,
  - $y^{(n)} = a^n \sin(ax + \frac{n\pi}{2})$
  - If $y = \sin 3x$, then $y^{(5)} = 243\cos 3x$

# Higher Order Differential Coefficients as Series (2/4)

- If $y = \cos ax$,
  - $y^{(n)} = a^n \cos(ax + \frac{n\pi}{2})$
  - If $y = 4\cos 2x$, then $y^{(6)} = -256\cos 2x$
- If $y = x^a$,
  - $y^{(n)} = \frac{a!}{(a-n)!} x^{a-n}$
  - If $y = 2x^6$, then $y^{(4)} = 720x^2$

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# Higher Order Differential Coefficients as Series (3/4)

- If $y = \sinh ax$,
  - $y^{(n)} = \frac{a^n}{2}\{[1+(-1)^n]\sinh ax + [1-(-1)^n]\cosh ax\}$
  - If $y = \sinh 2x$, then $y^{(5)} = 32\cosh 2x$
- If $y = \cosh ax$,
  - $y^{(n)} = \frac{a^n}{2}\{[1-(-1)^n]\sinh ax + [1+(-1)^n]\cosh ax\}$
  - If $y = (\cosh 3x)/9$, then $y^{(7)} = 243\sinh 3x$

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# Higher Order Differential Coefficients as Series (4/4)

- If $y = \ln ax$,
  - $y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}$
  - If $y = \ln 5x$, then $y^{(6)} = -120/x^6$
  - $(0)! = 1$

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# Exercise 194

Determine the following derivatives:

- **Exercise 3.** $y^{(9)}$ when $y = 3\cos(2t/3)$
  $[-2^9\sin(2t/3)/3^8]$
- **Exercise 5.** $y^{(7)}$ when $y = (\sinh 2x)/4$
  [32cosh 2x]
- **Exercise 7.** $y^{(7)}$ when $y = (\ln 2t)/3$
  $[240/t^7]$

# Leibniz's Theorem (1/3)

- If $y = uv$ where $u$ and $v$ are each functions of $x$, then by using the product rule,

$$
\begin{aligned}
y' &= u'v + uv' \\
y'' &= u''v + u'v' + u'v' + uv'' \\
&= u''v + 2u'v' + uv'' \\
y''' &= u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv''' \\
&= u'''v + 3u''v' + 3u'v'' + uv''' \\
y^{(4)} &= u^{(4)}v + u'''v' + 3u'''v' + 3u''v'' + 3u''v'' + 3u'v''' + u'v''' + uv^{(4)} \\
&= u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}
$$

# Leibniz's Theorem (2/3)

From above equations, it is seen that

1. The $n$'th derivative of $u$ decreases by 1 moving from left to right
2. The $n$'th derivative of $v$ increases by 1 moving from left to right
3. The coefficients 1, 4, 6, 4, 1 are the normal binomial coefficients

In fact, $(uv)^{(n)}$ may be obtained by expanding $(u + v)^{(n)}$ using the binomial theorem, where the 'powers' are interpreted as derivatives.

# Leibniz's Theorem (3/3)

- Thus, expanding $(u + v)^{(n)}$ gives:

$$
\begin{aligned}
y^{(n)} = (uv)^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} \\
&+ \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} \\
&+ \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \ldots
\end{aligned}
$$

- This is a statement of Leibniz's theorem, which can be used to differentiate a product $n$ times.

# Problems

- **Problem 1.** Determine $y^{(n)}$ when $y = x^2e^{3x}$
  $[y^{(n)} = e^{3x}3^{n-2}(9x^2 + 6nx + n(n - 1))]$

- **Problem 2.** If $x^2y'' + 2xy' + y = 0$, show that: $x^2y^{(n+2)} + 2(n + 1)xy^{(n+1)} + (n^2 + n + 1)y^{(n)} = 0$

- **Problem 3.** Differentiate the following differential equation $n$ times: $(1 + x^2)y'' + 2xy' - 3y = 0$
  $[(1 + x^2)y^{(n+2)} + 2(n + 1)xy^{(n+1)}$
  $+ (n^2 + n - 3)y^{(n)} = 0]$

# Problems & Exercise 195

- **Problem 4.** Find the 5th derivative of $y = x^4 \sin x$
  $[y^{(5)} = (x^4 - 120x^2 + 120)\cos x$
  $+ (20x^3 - 240x)\sin x]$
- **Exercise 6.** If $y = x^5 \ln 2x$ find $y^{(3)}$.
  $[y^{(3)} = x^2(47 + 60\ln 2x)]$
- **Exercise 7.** Given $2x^2y'' + xy' + 3y = 0$ show that $2x^2y^{(n+2)} + (4n + 1)xy^{(n+1)} + (2n^2 - n + 3)y^{(n)} = 0$

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# Power Series Solution by The Leibniz-Maclaurin Method

- For second order differential equations that cannot be solved by algebraic methods, the Leibniz-Maclaurin method produces a solution in the form of infinite series of powers of the unknown variable.

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# 5-Step Procedure

- Differentiate the given equation $n$ times, using the Leibniz theorem (equation)
- Rearrange the result to obtain the recurrence relation (or recurrence formula) at $x = 0$
- Determine the values of the derivatives at $x = 0$, i.e., find $(y)_0$ and $(y')_0$
- Substitute in the Maclaurin expansion for $y = f(x)$
- Simplify the result where possible and apply boundary condition (if given)

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# Problems

- **Problem 5.** Determine the power series solution of the differential equation: $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0$ using Leibniz-Maclaurin's method, given the boundary conditions that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

$$[y = \left\{1 - \frac{x^2}{1} + \frac{x^4}{1\times3} - \frac{x^6}{3\times5} + \frac{x^8}{3\times5\times7} - \ldots\right\}$$

$$+ 2\left\{\frac{x}{1} - \frac{x^3}{1\times2} + \frac{x^5}{2\times4} - \frac{x^7}{2\times4\times6} + \ldots\right\}]$$

# Problems

- **Problem 6.** Determine the power series solution of the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$ given the boundary conditions that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$, using Leibniz-Maclaurin's method.

$$[y = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{3x^4}{4!} + \frac{6x^5}{5!} - \frac{10x^6}{6!} + \frac{25x^7}{7!} - \frac{61x^8}{8!} + \ldots]$$

# Exercise 196

- **Exercise 1.** Determine the power series solution of the differential equations: $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ using the Leibniz-Maclaurin method, given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

$$[y = (1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \frac{5 \times 9x^6}{6!} + \frac{5 \times 9 \times 13x^8}{8!} - \ldots) + 2(x - \frac{3x^3}{3!} + \frac{3 \times 7x^5}{5!} - \frac{3 \times 7 \times 11x^7}{7!} + \ldots)]$$

# Power Series Solution by the Frobenius Method

- A differential equation of the form $y'' + Py' + Qy = 0$, where $P$ and $Q$ are both functions of $x$, such that the equation can be represented by a power series, may be solved by the Frobenius method.

# 4-Step Procedure

- Assume a trial solution of the form $y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots\}$
- Differentiate the trial series
- Substitute the results in the given differential equation
- Equate coefficients of corresponding powers of the variable on each side of the equation; this enables index $c$ and coefficients $a_1$, $a_2$, $a_3$, … from the trial solution, to be determined.

# Problems

- **Problem 7.** Determine, using the Frobenius method, the general power series solution of the differential equation: $3x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}-y=0$
  - Let a trial solution be of the form $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \ldots + a_rx^{c+r} + \ldots$
  - Differentiating equation to obtain $y'$, $y''$
  - Substituting $y$, $y'$, $y''$ into each term of the given equation
  - The coefficients of each power of $x$ can be equate to zero.

# Problems

- The coefficient of $x^{c-1}$ is equate to zero: $a_0c(3c - 2) = 0$ (indicial equation)
- The coefficient of $x^c$ is equate to zero: $a_1(3c + 1)(c + 1) - a_0 = 0$
- A general relationship can be obtained for $x^{c+r}$, where $r \geq 0$: $a_{r+1}\{(c + r + 1)(3c + 3r + 1)\} - a_r = 0$
- From indicial equation $c = 0$ or $c = 2/3$ (since $a_0 \neq 0$).
  - When $c = 0$: $y=a_0\{1+x+\dfrac{x^2}{(2\times4)}+\dfrac{x^3}{(2\times3)(4\times7)}+\dfrac{x^4}{(2\times3\times4)(4\times7\times10)}+\ldots\}$
  - When $c = 2/3$: $y=a_0x^{2/3}\{1+\dfrac{x}{5}+\dfrac{x^2}{(2\times5\times8)}+\dfrac{x^3}{(2\times3)(5\times8\times11)}+\dfrac{x^4}{(2\times3\times4)(5\times8\times11\times14)}+\ldots\}$

# Problems

- **Problem 7. (cont.)**

$$[y=A\{1+x+\frac{x^2}{(2\times4)}+\frac{x^3}{(2\times3)(4\times7)}+\frac{x^4}{(2\times3\times4)(4\times7\times10)}+\ldots\}$$
$$+Bx^{2/3}\{1+\frac{x}{5}+\frac{x^2}{(2\times5\times8)}+\frac{x^3}{(2\times3)(5\times8\times11)}$$
$$+\frac{x^4}{(2\times3\times4)(5\times8\times11\times14)}+\ldots\}]$$

# Problems

- **Problem 9.** Use the Frobenius method to determine the general power series solution of the differential equation: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y=0$
  - Let a trial solution be of the form $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \ldots + a_rx^{c+r} + \ldots$
  - Differentiating equation to obtain $y'$, $y''$
  - Substituting $y$ and $y''$ into each term of the given equation
  - The indicial equation is obtained by equating the coefficient of the lowest power of $x$ to zero: $a_0c(c - 1) = 0$, $c = 0$ or $c = 1$, since $a_0 \neq 0$

## Problems

- For the term in $x^{c-1}$, i.e., $a_1 c(c+1) = 0$
  - With $c = 1$, $a_1 = 0$
  - When $c = 0$, $a_1$ is indeterminate
- For the term in $x^{c}$, $a_2 = 2a_0/[(c+1)(c+2)]$
- For the term in $x^{c+r}$, $a_{r+2} = 2a_r/[(c+r+1)(c+r+2)]$
- When $c = 0$:

$$y = P\{1 + \frac{2x^2}{2!} + \frac{4x^4}{4!} + \ldots\} + Q\{x + \frac{2x^3}{3!} + \frac{4x^5}{5!} + \ldots\}$$

- When $c = 1$:

$$y = K\{x + \frac{2x^3}{3!} + \frac{4x^5}{5!} + \ldots\}$$

$$[y = P\{1 + \frac{2x^2}{2!} + \frac{4x^4}{4!} + \ldots\} + Q\{x + \frac{2x^3}{3!} + \frac{4x^5}{5!} + \ldots\}]$$

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## Exercise 197

- **Exercise 1.** Produce, using Frobenius' method, a power series solution for the differential equation: $2x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

$$[y = A\{1 + x + \frac{x^2}{(2\times 3)} + \frac{x^3}{(2\times 3)(3\times 5)} + \ldots\}$$
$$+ Bx^{1/2}\{1 + \frac{x}{(1\times 3)} + \frac{x^2}{(1\times 2)(3\times 5)}$$
$$+ \frac{x^3}{(1\times 2\times 3)(3\times 5\times 7)} + \ldots\}]$$

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## Exercise 197

- **Exercise 2.** Use the Frobenius method to determine the general power series solution of the differential equation: $\frac{d^2y}{dx^2} + y = 0$

$$[y = A\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\} + B\{x - \frac{x}{3!} + \frac{x^5}{5!} - \ldots\}$$
$$= P\cos x + Q\sin x]$$

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## Bessel's Equation and Bessel's Functions

- Bessel's equation is of the form:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$$

where $v$ is a real constant.

- The equation, which has applications in electric fields, vibrations and heat conduction, may be solved using Frobenius' method.

# Problems

- **Problem 10.** Determine the general power series solution of Bessel's equation.
  - Rewritten Bessel's equation as: $x^2y'' + xy' + (x^2 - v^2)y = 0$
  - Let a trial solution be of the form $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \ldots + a_rx^{c+r} + \ldots$
  - Differentiating equation to obtain $y'$, $y''$
  - Substituting $y$, $y'$, $y''$ into each term of the given equation
  - The indicial equation is obtained by equating the coefficient of the lowest power of $x$ to zero: $a_0\,(c^2 - v^2) = 0$, $c = +v$ or $c = -v$, since $a_0 \neq 0$

# Problems

- For the term in $x^{c+r}$, the recurrence relation is:

$$a_r = \frac{a_{r-2}}{v^2 - (c+r)^2} \quad \text{for } r \geq 2$$

- For the term in $x^{c+1}$,
  - If $c = v$, $a_1(2v + 1) = 0$
  - If $c = -v$, $a_1(1 - 2v) = 0$
  - Since $v$ is a real constant, $a_1 = 0$
- Since $a_1 = 0$, $a_3 = a_5 = a_7 = \ldots = 0$
- When $c = +v$,
  - $a_2 = -a_0/[2^2(v + 1)]$
  - $a_4 = a_0/[2^4{\cdot}2(v + 1)(v + 2)]$
  - $a_6 = -a_0/[2^6{\cdot}3!(v + 1)\,(v + 2)\,(v + 3)]$

# Problems

- The resulting solution for $c = +v$ is given by:

$$y = u = Ax^v\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)}$$

$$- \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\}$$

  which is valid provided $v$ is not a negative integer and where $A$ is an arbitrary constant.
- When $c = -v$,
  - $a_2 = \boxed{a_0/[2^2(v - 1)]}$
  - $a_4 = a_0/[2^4{\cdot}2(v - 1)(v - 2)]$
  - $a_6 = a_0/[2^6{\cdot}3!(v - 1)\,(v - 2)\,(v - 3)]$

# Problems

- The resulting solution for $c = -v$ is given by:

$$y = w = Bx^{-v}\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)}$$

$$+ \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\}$$

  which is valid provided $v$ is not a positive integer and where $B$ is an arbitrary constant.

# Problems

- **Problem 10. (cont.)**

$$[y = u + w = Ax^{v}\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)}$$

$$- \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\}$$

$$+ Bx^{-v}\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)}$$

$$+ \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\}]$$

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# The Gamma Function

- The solution of the Bessel equation of **Problem 10** may be expressed in terms of gamma functions.
- The gamma function $\Gamma(x)$ is defined by the integral $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ and is convergent for $x > 0$
  - From the definition and by using integration by parts $\Gamma(x+1) = x\Gamma(x)$
  - Similarly, $\Gamma(x+2) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$

$$\Gamma(x+3) = (x+2)\Gamma(x+2) = (x+2)(x+1)x\Gamma(x)$$

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# Bessel Functions

- The power series solution of the Bessel equation may be written in terms of gamma functions.
- **Problem 11.** Show that the power series solution of the Bessel equation of worked problem 10 may be written in terms of the Bessel functions $J_v(x)$ and $J_{-v}(x)$ as:

$$AJ_v(x) + BJ_{-v}(x)$$

$$= A\left(\frac{x}{2}\right)^{v}\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\boxed{\Gamma(v+3)}} - \ldots\}$$

$$+ B\left(\frac{x}{2}\right)^{-v}\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2^2(1!)\Gamma(2-v)} + \frac{x^4}{2^4(2!)\Gamma(3-v)} - \ldots\}$$

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# Problems

- **Problem 11. (cont.)**
  - The power series solution of the Bessel equation:

$$y = Ax^{v}\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\}$$

$$+ Bx^{-v}\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\}$$

  - Written in terms of the Bessel functions $J_v(x)$ and $J_{-v}(x)$

$$AJ_v(x) + BJ_{-v}(x)$$

$$= A\left(\frac{x}{2}\right)^{v}\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \ldots\}$$

$$+ B\left(\frac{x}{2}\right)^{-v}\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2^2(1!)\Gamma(2-v)} + \frac{x^4}{2^4(2!)\Gamma(3-v)} - \ldots\}$$

# Problems

- **Problem 11. (cont.)**
  - From Problem 10, when $c = +v$

$$a_2 = \frac{-a_0}{2^2(v+1)},\quad a_r = \frac{a_{r-2}}{v^2-(c+r)^2} = \frac{-a_{r-2}}{r^2+2vr}$$

  - If we let $a_0 = \dfrac{1}{2^v\Gamma(v+1)}$

then $a_2 = \dfrac{-1}{2^{v+2}\Gamma(v+2)},\quad a_4 = \dfrac{a_2}{-4(2v+4)} = \dfrac{1}{2^{v+4}(2!)\Gamma(v+3)}$

  - The recurrence relation is:

$$a_r = \frac{(-1)^{r/2}}{2^{v+r}(\frac{r}{2}!)\Gamma(v+\frac{r}{2}+1)}$$

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# Problems

- **Problem 11. (cont.)**
  - And if we let $r = 2k$, then: for $k = 1, 2, 3, \ldots$

$$a_{2k} = \frac{(-1)^k}{2^{v+2k}(k!)\Gamma(v+k+1)}$$

  - Hence, it is possible to write the new form for equation $y = u = Ax^v\{a_0 + a_2x^2 + a_4x^4 + \ldots\}$ as:

$$y = Ax^v\{\frac{1}{2^v\Gamma(v+1)} - \frac{x^2}{2^{v+2}(1!)\Gamma(v+2)} + \frac{x^4}{2^{v+4}(2!)\Gamma(v+3)} - \ldots\}$$

  - This is called the Bessel function of the first order kind, of order $v$, is denoted by $J_v(x)$, (provided $v$ is not a negative integer)

$$J_v(x) = \left(\frac{x}{2}\right)^v\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \ldots\}$$

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# Problems

- **Problem 11. (cont.)**
  - When $c = -v$, replacing $v$ by $-v$ in the recurrence relation gives:

$$a_{2k} = \frac{(-1)^k}{2^{2k-v}(k!)\Gamma(k-v+1)}$$

  - When $k = 0$, $a_0 = \dfrac{1}{2^{-v}\Gamma(1-v)}$
  - When $k = 1$, $a_2 = \dfrac{-1}{2^{2-v}(1!)\Gamma(2-v)}$
  - When $k = 2$, $a_4 = \dfrac{1}{2^{4-v}(2!)\Gamma(3-v)}$

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# Problems

- **Problem 11. (cont.)**
  - Hence, $y = Bx^{-v}\{\dfrac{1}{2^{-v}\Gamma(1-v)} - \dfrac{x^2}{2^{2-v}(1!)\Gamma(2-v)} + \dfrac{x^4}{2^{4-v}(2!)\Gamma(3-v)} - \ldots\}$

i.e., $J_{-v}(x) = \left(\dfrac{x}{2}\right)^{-v}\{\dfrac{1}{\Gamma(1-v)} - \dfrac{x^2}{2^2(1!)\Gamma(2-v)} + \dfrac{x^4}{2^4(2!)\Gamma(3-v)} - \ldots\}$

provided $v$ is not a positive integer.

$$\begin{aligned} y &= AJ_v(x) + BJ_{-v}(x) \\ &= A\left(\frac{x}{2}\right)^v\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+3)} - \ldots\} \\ &+ B\left(\frac{x}{2}\right)^{-v}\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2^2(1!)\Gamma(2-v)} + \frac{x^4}{2^4(2!)\Gamma(3-v)} - \ldots\} \end{aligned}$$

# Another Bessel Function

- It may be shown that another series for $J_n(x)$ is given by:

$$J_n(x) = \left(\frac{x}{2}\right)^n \{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \ldots\}$$

- From this series, two commonly used function are derived.

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \ldots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \ldots$$

# Exercise 198

- **Exercise 1.** Determine the power series solution of the Bessel function: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$ when $v = 2$, up to and including the term in $x^6$.

  [$y = Ax^2\{1 - x^2/12 + x^4/384 - \ldots\}$]

- **Exercise 3.** Evaluate the Bessel functions $J_0(x)$ and $J_1(x)$ when $x = 1$, correct to 3 decimal places.

  [$J_0(x) = 0.765$, $J_1(x) = 0.440$]

# Legendre's Equation and Legendre Polynomials

- Another important differential equation in physics and engineering applications is Legendre's equation of the form:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0$$

where $k$ is a real constant.

# Problems

- **Problem 12.** Determine the general power series solution of Legendre's equation.

  To solve Legendre's equation $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$ using the Frobenius method.
  - Let a trial solution be of the form $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \ldots + a_rx^{c+r} + \ldots$
  - Differentiating equation to obtain $y'$ and $y''$
  - Substituting $y$, $y'$, $y''$ into each term of the given equation

# Problems

- **Problem 12. (cont.)**
  - The indicial equation is obtained by equating the coefficient of the lowest power of $x$ to zero: $a_0c(c-1) = 0$, $c = 0$ or $c = 1$, since $a_0 \neq 0$
  - For the term in $x^{c-1}$, $a_1c(c+1) = 0$
    - If $c = 1$, $a_1 = 0$
    - If $c = 0$, $a_1$ is indeterminate
  - For the term in $x^{c+r}$,

$$a_{r+2} = \frac{a_r[(c+r)(c+r+1)-k(k+1)]}{(c+r+1)(c+r+2)}$$

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# Problems

- **Problem 12. (cont.)**
  - When $c = 0$, $a_{r+2} = \dfrac{a_r[r(r+1)-k(k+1)]}{(r+1)(r+2)}$

$$a_2 = \frac{a_0[-k(k+1)]}{(1)(2)},\quad a_3 = \frac{-a_1(k-1)(k+2)}{3!}$$

$$a_4 = \frac{a_0k(k+1)(k+3)(k-2)}{4!},\quad a_5 = \frac{a_1(k-1)(k-3)(k+2)(k+4)}{5!}$$

  - Hence,

$$y = a_0\left\{1-\frac{k(k+1)}{2!}x^2+\frac{k(k+1)(k-2)(k+3)}{4!}x^4-\ldots\right\}$$
$$+a_1\left\{x-\frac{(k-1)(k+2)}{3!}x^3+\frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5-\ldots\right\}$$

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# Problems

- **Problem 12. (cont.)**
  - If two solutions of the indicial equation differ by an integer (e.g., $c = 0$ and $1$) and if one coefficient is indeterminate (as with when $c = 0$), then the complete solution is always given by using this value of $c$.

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# Legendre's Polynomials

$$y = a_0\left\{1-\frac{k(k+1)}{2!}x^2+\frac{k(k+1)(k-2)(k+3)}{4!}x^4-\ldots\right\}$$
$$+a_1\left\{x-\frac{(k-1)(k+2)}{3!}x^3+\frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5-\ldots\right\}$$

- When $k$ in the general power series solution of Legendre's equation is an integer, say $n$, one of the solution series terminates after a finite number of terms.
- The resulting polynomial in $x$, denoted by $P_n(x)$, is called a Legendre polynomial.
- Constants $a_0$ and $a_1$ are chosen so that $y = 1$ when $x = 1$.

# Problems

- **Problem 13.** Determine the Legendre polynomial $P_2(x)$.
  [$P_2(x) = (3x^2 - 1)/2$]
- **Problem 14.** Determine the Legendre polynomial $P_3(x)$.
  [$P_3(x) = (5x^3 - 3x)/2$]

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# Rodrigue's Formula & Problems

- An alternative method of determining Legendre polynomials is by using Rodrigue's formula, which states:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}$$

- **Problem 15.** Determine the Legendre polynomial $P_2(x)$ using Rodrigue's formula.
  [$P_2(x) = (3x^2 - 1)/2$]

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# Problems & Exercise 199

- **Problem 16.** Determine the Legendre polynomial $P_3(x)$ using Rodrigue's formula.
  [$P_3(x) = (5x^3 - 3x)/2$]
- **Exercise 1.** Determine the power series solution of the Legendre equation: $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$ when $k = 2$, up to and including the term in $x^5$.
  [$y = a_0\{1 - 3x^2\} + a_1\{x - 2x^3/3 - x^5/5\}$]
- **Exercise 2.** Find the Legendre polynomial $P_4(x)$.
  [$(35x^4 - 30x^2 + 3)/8$]

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- Bessel's equation is of the form: $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - v^2)y = 0$
- The power series solution of the Bessel equation:

$$y = Ax^v\{1 - \frac{1}{1!(1+v)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+v)(2+v)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!(1+v)(2+v)(3+v)}\left(\frac{x}{2}\right)^6 + \ldots\}$$
$$+ Bx^{-v}\{1 - \frac{1}{1!(1-v)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-v)(2-v)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!(1-v)(2-v)(3-v)}\left(\frac{x}{2}\right)^6 + \ldots\}$$

- Written in terms of the Bessel functions $J_v(x)$ and $J_{-v}(x)$

$$y = AJ_v(x) + BJ_{-v}(x)$$
$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{i=0}^{\infty} \frac{(-1)^i}{(i!)\Gamma(i+1+v)}\left(\frac{x}{2}\right)^{2i}; \; J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{i=0}^{\infty} \frac{(-1)^i}{(i!)\Gamma(i+1-v)}\left(\frac{x}{2}\right)^{2i}$$

- Another Bessel function

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{i=0}^{\infty} \frac{(-1)^i}{(i!)(n+i)!}\left(\frac{x}{2}\right)^{2i}$$

$$\Gamma(n) = (n-1)!$$
$$\because \Gamma(x+1) = x\Gamma(x), \Gamma(1) = \int_0^{\infty} t^0 e^{-t}\mathrm{d}t = -e^{-t}\Big|_0^{\infty} = 1$$

- Legendre’s equation is of the form:

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+k(k+1)y=0$$

- The power series solution of the Legendre equation:

$$\begin{aligned} y = a_0\{&\frac{x^0}{0!}-\frac{x^2(k-0)(k+1)}{2!}+\frac{x^4(k-0)(k+1)(k-2)(k+3)}{4!}-...\} \\ +a_1\{&\frac{x^1}{1!}-\frac{x^3(k-1)(k+2)}{3!}+\frac{x^5(k-1)(k+2)(k-3)(k+4)}{5!}-...\} \end{aligned}$$

- Rodrigue’s formula

$$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}$$