Chapter 51: Second Order Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

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2008 Spring

Outline (1/2)

- Complementary Function and Particular Integral
- Procedure to Solve Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$
- Worked Problems on Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a constant or polynomial
- Worked Problems on Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is an exponential function

Outline (2/2)

- Worked Problems on Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a sine or cosine function
- Worked Problems on Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a sum or a product

Complementary Function and Particular Integral (1/3)

- If in the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad (1)$$

the substitution $y = u + v$ is made then:

$$a\frac{d^2(u+v)}{dx^2}+b\frac{d(u+v)}{dx}+c(u+v)=f(x)$$

- Rearrange gives:

$$(a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu)+(a\frac{d^2v}{dx^2}+b\frac{dv}{dx}+cv)=f(x)$$

Complementary Function and Particular Integral (2/3)

- If we let

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+b\frac{\mathrm{d}v}{\mathrm{d}x}+cv=f(x) \qquad (2)$$

then

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+b\frac{\mathrm{d}u}{\mathrm{d}x}+cu=0 \qquad (3)$$

- The general solution, u, of equation (3) will contain two unknown constants, as required for the general solution of equation (1).

Complementary Function and Particular Integral (3/3)

- The method of solution of equation (3) is shown in Chapter 50. The function u is called the complementary function (C.F.).
- If the particular solution, v, of equation (2) can be determined without containing any unknown constants then $y = u + v$ will give the general solution of equation (1).
- The function v is called the particular integral (P.I.).
- Hence the general solution of equation (1) is given be: y = C.F. + P.I.

Procedure (1/2)

- Rewrite the given differential equation as $(a\mathrm{D}^2 + b\mathrm{D} + c)y = f(x)$.
- Substitute m for D, and solve the auxiliary equation $am^2 + bm + c = 0$ for m.
- Obtain the complementary function u.
- To determine the particular integral, v, firstly assume a particular integral which is suggested by $f(x)$, but which contains undetermined coefficients.

Procedure (2/2)

- Substitute the suggested P.I. into the differential equation $(a\mathrm{D}^2 + b\mathrm{D} + c)v = f(x)$ and equate relevant coefficients to find the constants introduced.
- The general solution is given by y = C.F. + P.I., i.e., $y = u + v$.
- Given boundary conditions, arbitrary constants in the C.F. may be determined and the particular solution of the differential equation obtained.

Type	*Straightforward cases* Try as particular integral:	'*Snag*' cases Try as particular integral:	*See problem*
$f(x) = C$	$v = k$	$v = kx$ (used when C.F. contains a constant)	1, 2
$f(x) = L + Mx + Nx^2 + \ldots$	$v = a + bx + cx^2 + \ldots$		3
$f(x) = Ae^{ax}$	$v = ke^{ax}$	• $v = kxe^{ax}$ (used when e^{ax} appears in the C.F.) • $v = kx^2e^{ax}$ (used when e^{ax} **and** xe^{ax} both appear in the C.F.)	4, 5, 6
$f(x) = a \sin px + b \cos px$	$v = A \sin px + B \cos px$	$v = x(A \sin px + B \cos px)$ (used when $\sin px$ and/or $\cos px$ appears in the C.F.)	7, 8
$f(x)$ = a sum, e.g., • $f(x) = 4x^2 - 3\sin 2x$ • $f(x) = 2 - x + e^{3x}$	• $v = ax^2 + bx + c + d \sin 2x + e \cos 2x$ • $v = ax + b + ce^{3x}$		9
$f(x)$ = a product, e.g., $f(x) = 2e^x \cos 2x$	$v = e^x(A \sin 2x + B \cos 2x)$		10

Worked Problems – $f(x)$: A Constant or Polynomial

- **Problem 1.** Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4.$
  [y = Ae^{x} + Be^{-2x} - 2]
- **Problem 2.** Determine the particular solution of the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 9$, given the boundary conditions that when x = 0, y = 0 and $\frac{dy}{dx} = 0.$
  [y = e^{3x} - 3x - 1]

Worked Problems – $f(x)$: A Constant or Polynomial & Exercise 190

- **Problem 3.** Solve the differential equation $2\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 12y = 3x - 2.$
  [y = Ae$^{3x/2}$ + Be^{4x} + x/4 + 1/16]

Find the particular solution of the given differential equation.

- **Exercise 3.** $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 4y = 8$; when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0.$
  [y = 2(3e$^{-4x/3}$ + 4e^{x})/7 - 2]

Exercise 190

- **Exercise 5.** The charge q in an electric circuit at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E$ where L, R, C, and E are constants. Solve the equation given L = 2 H, C = 200 x 10^{-6} F, and E = 250 V, when (a) R = 200 Ω and (b) R is negligible. Assume that when $t = 0$, $q = 0$, and $\frac{dq}{dt} = 0.$
  [(a) q = 1/20 - (5t/2 + 1/20)e^{-50t}
  (b) q = (1 - cos 50t)/20]

Worked Problems – $f(x)$: An Exponential Function

- **Problem 4.** Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^{4x}$ given the boundary conditions that when $x = 0$, $y = -2/3$ and $\frac{dy}{dx} = 4\frac{1}{3}$

  [y = (4x – 1)e^x + e^{4x}/3]

- **Problem 5.** Solve the differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 5e^{3x/2}$.

  [y = A$e^{3x/2}$ + Be^{-x} + x$e^{3x/2}$]

Worked Problems – $f(x)$: An Exponential Function & Exercise 191

- **Problem 6.** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3e^{2x}$.

  [y = (Ax + B)e^{2x} + 3x^2e^{2x}/2]

Find the general solution of the give differential equations.

- **Exercise 2.** $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{-x}$

  [y = Ae^{4x} + Be^{-x} - 3xe^{-x}/5]

Exercise 191

- **Exercise 4.** $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 12e^{t/3}$

  [y = (At + B)$e^{t/3}$ + 2$t^2e^{t/3}$/3]

Find the particular solutions of the given differential equation.

- **Exercise 5.** $5\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 2y = 3e^x$; when $x = 0$, $y = 1/4$ and $\frac{dy}{dx} = 0$.

  [y = 5(e^{-2x} – $e^{x/5}$)/44 + e^x/4]

Worked Problems – $f(x)$: A Sine or Cosine Function

- **Problem 7.** Solve the differential equation $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 5y = 6\sin 2x$.

  [y = Ae^x + B$e^{-5x/2}$
  – 2(39 sin 2x + 18 cos 2x)/205]

- **Problem 8.** Solve $\frac{d^2y}{dx^2} + 16y = 10\cos 4x$ given $y = 3$ and $\frac{dy}{dx} = 4$ when $x = 0$.

  [y = 3 cos 4x + sin 4x + 5x sin 4x/4]

Exercise 192

- **Exercise 3.** Find the general solution of the given differential equation: $\frac{d^2y}{dx^2} + y = 4\cos x$

  [y = A cos x + B sin x + 2x sin x]

Exercise 192

- **Exercise 5.** A differential equation representing the motion of a body is $\frac{d^2y}{dt^2} + n^2y = k\sin pt$, where k, n and p are constants. Solve the equation (given $n \neq 0$ and $p^2 \neq n^2$) given that when $t = 0$, $\frac{dy}{dt} = 0$.

  $[y = \frac{k}{n^2 - p^2}(\sin pt - \frac{p}{n}\sin nt)]$

Worked Problems – $f(x)$: A Sum or A Product

- **Problem 9.** Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 12x - 50\sin x$.

  [y = $Ae^{2x} + Be^{-3x} - 2x - 1/3 + 7\sin x + \cos x$]

- **Problem 10.** Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 3e^x\cos 2x$, given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 3$.

  [y = $e^x(3\cos x + \sin x) - e^x\cos 2x$]

Exercise 193

Find the particular solutions of the given differential equations.

- **Exercise 5.** $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x} + 20$; when $x = 0$, $y = 0$ and $\frac{dy}{dx} = -\frac{1}{3}$.

  [y = $4e^{5x}/3 - 10e^{2x}/3 - xe^{2x}/3 + 2$]

Exercise 193

- **Exercise 6.** $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 6e^x \cos x$; when $x = 0$, $y = -\frac{21}{29}$ and $\frac{dy}{dx} = -7\frac{12}{29}$

  [$y = 2e^{-3x/2} - 2e^{2x} + 3e^x(3\sin x - 7\cos x)/29$]