Chapter 50: Second Order Differential Equations of the Form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

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Outline

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Introduction (1/2)

- An equation of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, where a, b, and c are constants, is called a linear second order differential equation with constant coefficients.
  - Homogeneous Differential Equation
  - Non-Homogeneous Differential Equation
- 'D-operator' form: If D represents $\frac{d}{dx}$ and D^2 represents $\frac{d^2}{dx^2}$ then the above equation may be stated as $(a\mathrm{D}^2+b\mathrm{D}+c)y=0$.

Introduction (2/2)

- If $y=Ae^{mx}$ then $\frac{dy}{dx}=Ame^{mx}$ and $\frac{d^2y}{dx^2}=Am^2e^{mx}$.
- Substitute these values into $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ gives:
  $a(Am^2e^{mx})+b(Ame^{mx})+c(Ae^{mx})=0$
  i.e., $Ae^{mx}(am^2+bm+c)=0$
- Thus $y=Ae^{mx}$ is a solution of the given equation provided that $(am^2+bm+c)=0$.
- $am^2+bm+c=0$ is called the auxiliary equation.

Procedure (1/2)

- Rewrite the differential equation
$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$
as $(aD^2+bD+c)y=0$
- Substitute m for D and solve the auxiliary equation $am^2+bm+c=0$ for m.
- If the roots of the auxiliary equation are:
  - Real and different, say $m=\alpha$ and $m=\beta$, then the general solution is
$$y=Ae^{\alpha x}+Be^{\beta x}$$

Procedure (2/2)

- Real and equal, say $m=\alpha$ twice, then the general solution is
$$y=(Ax+B)e^{\alpha x}$$
- Complex, say $m=\alpha\pm j\beta$, then the general solution is
$$y=e^{\alpha x}\{A\cos\beta x+B\sin\beta x\}$$

- Given boundary conditions, constants A and B, may be determined and the particular solution of the differential equation obtained.

Worked Problems (1/3)

- **Problem 1.** Determine the general solution of $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$. Find also the particular solution given that when $x=0$, $y=4$ and $\frac{dy}{dx}=9$.
[general solution: $y = Ae^{x/2} + Be^{-3x}$
particular solution: $y = 6e^{x/2} - 2e^{-3x}$]

Worked Problems (2/3)

- **Problem 2.** Find the general solution of $9\frac{d^2y}{dt^2}-24\frac{dy}{dt}+16y=0$ and also the particular solution given the boundary conditions that when $t=0$, $y=\frac{dy}{dt}=3$.
[general solution: $y = (At+B)e^{4t/3}$
particular solution: $y = (3 - t)e^{4t/3}$]

Worked Problems (3/3)

- **Problem 3.** Solve the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$, given that when $x = 0$, $y = 3$ and $\frac{dy}{dx} = 7$.

  [general solution: $y = e^{-3x}(A\cos 2x + B\sin 2x)$
  particular solution: $y = e^{-3x}(3\cos 2x + 8\sin 2x)$]

Exercise 188

Find the particular solution of the given differential equations for the stated boundary conditions.

- **Exercise 4.** $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$; when $x = 0$, $y = 5$ and $\frac{dy}{dx} = -1$.

  [$y = 3e^{2x/3} + 2e^{-3x/2}$]
- **Exercise 7.** $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 9x = 0$; when $t = 0$, $x = 2$ and $\frac{dx}{dt} = 0$.

  [$x = 2(1 - 3t)e^{3t}$]

Exercise 188

- **Exercise 8.** $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$; when $x = 0$, $y = 4$ and $\frac{dy}{dx} = 0$.

  [$y = 2e^{-3x}(2\cos 2x + 3\sin 2x)$]

Further Worked Problems

- **Problem 5.** Given the differential equation $\frac{d^2V}{dt^2} = \omega^2 V$, where ω is a constant, show that its solution may be expressed as:

  $$V = 7\cosh \omega t + 3\sinh \omega t$$

  given the boundary conditions that when $t = 0$, $V = 7$ and $\frac{dV}{dt} = 3\omega$.

Further Worked Problems

- **Problem 7.** The oscillations of a heavily damped pendulum satisfy the differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$, where x cm is the displacement of the bob at time t seconds. The initial displacement is equal to $+4$ cm and the initial velocity (i.e., $\frac{dx}{dt}$) is 8 cm/s. Solve the equation for x.

  [x = 4(3e^{-2t} – 2e^{-4t}) cm]

Exercise 189

- **Exercise 1.** The charge, q, on a capacitor in a certain electrical circuit satisfies the differential equation $\frac{d^2q}{dt^2}+4\frac{dq}{dt}+5q=0$. Initially (i.e., when t $=0$), $q=Q$ and $\frac{dq}{dt}=0$. Show that the charge in the circuit can be expressed as:

$$q=\sqrt{5}Qe^{-2t}\sin(t+0.464)$$

If $R\sin(\omega t+\alpha)=a\sin\omega t+b\cos\omega t$

then $a=R\cos\alpha, b=R\sin\alpha$

$R=\sqrt{(a^2+b^2)}, \alpha=\tan^{-1}(b/a)$

Exercise 189

- **Exercise 5.** $L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=0$ is an equation representing current i in an electric circuit. If inductance L is 0.25 henry, capacitance C is 29.76 x 10^{-6} farads and R is 250 ohms, solve the equation for i given the boundary conditions that when $t=0$, $i=0$ and $\frac{di}{dt}=34$.

  [i = (e^{-160t} – e^{-840t})/20]