Chapter 49: Numerical Methods for First Order Differential Equations

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Outline

- Introduction
- Euler's Method
- Worked Problems on Euler's Method
- An Improved Euler Method
- The Runge-Kutta Method

Introduction

- Not all first order differential equations may be solved by separating the variables (Chapter 46) or by the integrating factor method (Chapter 48).
- When a differential equation and known boundary conditions are given, an approximate solution may be obtained by applying a numerical method.

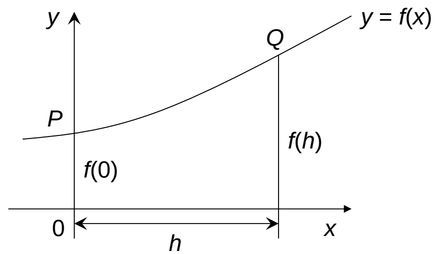
Euler's Method (1/4)

- From Chapter 8, Maclaurin's series may be stated as:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

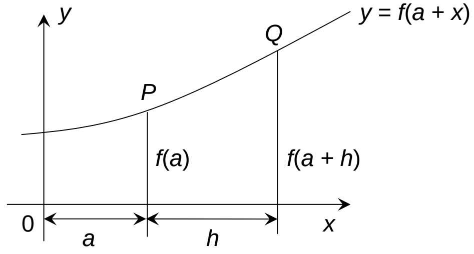
- Hence at some point $f(h)$:

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \ldots$$



Euler's Method (2/4)

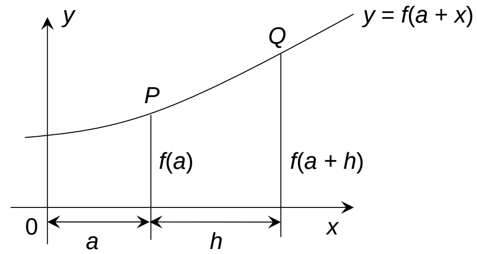
- If the y-axis and origin are moved a units to the left, the equation of the same curve relative to the new axis becomes $y = f(a + x)$ and the function value at P is $f(a)$.



Euler's Method (3/4)

- At point Q: $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$

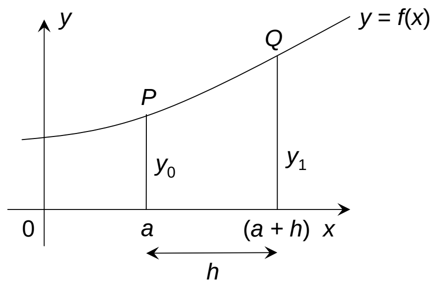
  which is a statement called Taylor's series.



Euler's Method (4/4)

- If h is the interval between two new ordinates y_0 and y_1, and if $f(a) = y_0$ and $y_1 = f(a + h)$, then Euler's method states: $f(a+h) = f(a) + hf'(a)$

  i.e., $y_1 = y_0 + h(y')_0$



Worked Problems on Euler's Method

- **Problem 1.** Obtain a numerical solution of the differential equation $dy/dx = 3(1 + x) - y$ given the initial conditions that x = 1 when y = 4, for the range x = 1.0 to x = 2.0 with intervals of 0.2. Draw the graph of the solution.

x_0	y_0	$(y')_0$
1	4	2
1.2	4.4	2.2
1.4	4.84	2.36
1.6	5.312	2.488
1.8	5.8096	2.5904
2.0	6.32768	

Worked Problems on Euler's Method

- **Problem 3.**
  (a) Obtain a numerical solution, using Euler's method, of the differential equation $dy/dx = y - x$, with the initial conditions that at x = 0, y = 2, for the range x = 0(0.1)0.5. Draw the graph of the solution.

x_0	y_0	$(y')_0$
0	2	2
0.1	2.2	2.1
0.2	2.41	2.21
0.3	2.631	2.331
0.4	2.8641	2.4641
0.5	3.11051	

Worked Problems on Euler's Method

- **Problem 3. (cont.)**
  (b) By an analytical method (using the integrating factor method), the solution of the above differential equation is given by $y = x + 1 + e^x$. Determine the percentage error at x = 0.3.
  [0.712%]

Exercise 185

- **Exercise 3.**
  (a) The differential equation $dy/dx + 1 = -y/x$ has the initial conditions that y = 1 at x = 2. Produce a numerical solution of the differential equation in the range x = 2.0(0.1)2.5.

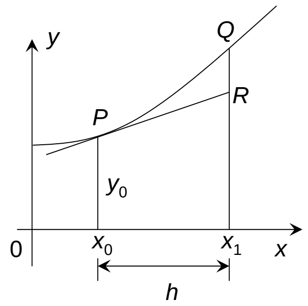
x_0	y_0
2.0	1
2.1	0.85
2.2	0.709524
2.3	0.577273
2.4	0.452174
2.5	0.333334

Exercise 185

- **Exercise 3. (cont.)**
  (b) If the solution of the differential equation by an analytical method is given by $y = 4/x - x/2$, determine the percentage error at x = 2.2.
  [1.206%]

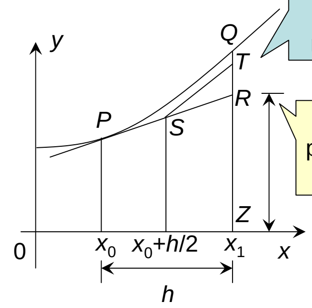
An Improved Euler Method

- In Euler's method, the gradient $(y')_0$ at $P_{(x0,\,y0)}$ across the whole interval h is used to obtain an approximate value of y_1 at point Q. QR is the resulting error in the result.



Euler-Cauchy Method (1/2)

- In Euler-Cauchy method, the gradient at $P_{(x0,\,y0)}$ across half the interval is used and then continues with a line whose gradient approximates to the gradient of the curve at x_1.



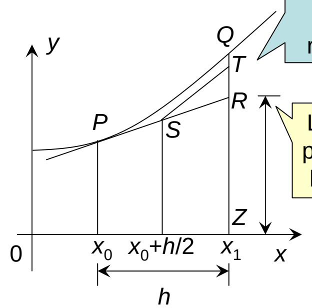
The error shown as QT is now less than the error QR used in the basic Euler method and the calculated results will be of greater accuracy.

Let yp_1 be the predicted value at point R using Euler's method, i.e., length RZ, where $yp_1 = y_0 + h(y')_0$

Euler-Cauchy Method (2/2)

- The corrected value, yc_1 in the improved Euler method is given by:

$$y_{C_1} = y_0 + \frac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$



The error shown as QT is now less than the error QR used in the basic Euler method and the calculated results will be of greater accuracy.

Let yp_1 be the predicted value at point R using Euler's method, i.e., length RZ, where $yp_1 = y_0 + h(y')_0$

Problems

- **Problem 4.** Apply the Euler-Cauchy method to solve the differential equation

$$\frac{dy}{dx} = y - x$$

in the range 0(0.1)0.5, given the initial conditions that at x = 0, y = 2.

x	y	y'
0	2	2
0.1	2.205	2.105
0.2	2.421025	2.221025
0.3	2.649232625	2.34923625
0.4	2.89090205	2.49090205
0.5	3.147446765	

Problems

- **Problem 4. (cont.)**

x	Euler method y	Euler-Cauchy method y	Exact value $y = x + 1 + e^x$	Error in Euler method	Error in Euler-Cauchy method
0	2	2	2	0	0
0.1	2.2	2.205	2.205170918	0.234%	0.00775%
0.2	2.41	2.421025	2.421402758	0.472%	0.0156%
0.3	2.631	2.649232625	2.649858808	0.712%	0.0236%
0.4	2.8641	2.89090205	2.891824698	0.959%	0.0319%
0.5	3.11051	3.147446765	3.148721271	1.214%	0.0405%

Exercise 186

- **Exercise 3.**
  (a) Apply the Euler-Cauchy method to solve the differential equation $dy/dx - x = y$ for the range x = 0 to x = 0.5 in increments of 0.1, given the initial conditions that when x = 0, y = 1.

x	y	y'
0	1	1
0.1	1.1	1.21
0.2	1.24205	1.44205
0.3	1.398465	1.698465
0.4	1.581804	1.981804
0.5	1.794893	

Exercise 186

- **Exercise 3. (cont.)**
  (b) The solution of the differential equation in part (a) is given by $y = 2e^x - x - 1$. Determine the percentage error, correct to 3 decimal places, when x = 0.4.
  [0.117%]

The Runge-Kutta Method (1/2)

- The 7 step procedure for the Runge-Kutta method, without proof, is as follows:
- To solve the differential equation $dy/dx = f(x, y)$ given the initial condition $y = y_0$ at $x = x_0$ for a range of values of $x = x_0(h)x_n$:

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$.
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f(x_n + h/2, y_n + hk_1/2)$
4. Evaluate $k_3 = f(x_n + h/2, y_n + hk_2/2)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$

The Runge-Kutta Method (2/2)

6. Use the values determined from steps 2 to 5 to evaluate: $y_{n+1} = y_n + h\{k_1 + 2k_2 + 2k_3 + k_4\}/6$
7. Repeat steps 2 to 6 for $n = 1, 2, 3, \ldots$

Problems

- **Problem 7.** Use the Runge-Kutta method to solve the differential equation: $dy/dx = y - x$ in the range 0(0.1)0.5, given the initial conditions that at x = 0, y = 2.

n	x_n	k_1	k_2	k_3	k_4	y_n
0	0					2
1	0.1	2.0	2.05	2.0525	2.10525	2.205171
2	0.2	2.105171	2.160430	2.163193	2.221490	2.421403
3	0.3	2.221403	2.282473	2.285527	2.349956	2.649859
4	0.4	2.349859	2.417339	2.420726	2.491932	2.891824
5	0.5	2.491824	2.566415	2.570145	2.648838	3.148720

Problems

- **Problem 7. (cont.)**

x	Euler's method y	Euler-Cauchy method y	Runge-Kutta method y	Exact value $y = x + 1 + e^x$
0	2	2	2	2
0.1	2.2	2.205	2.205171	2.205170918
0.2	2.41	2.421025	2.421403	2.421402758
0.3	2.631	2.649232625	2.649859	2.649858808
0.4	2.8641	2.89090205	2.891824	2.891824698
0.5	3.11051	3.147446765	3.148720	3.148721271

Exercise 187

- **Exercise 1.** Apply the Runge-Kutta method to solve the differential equation: $dy/dx = 3 - y/x$ for the range 1.0(0.1)1.5, given that the initial conditions that x = 1 when y = 2.

n	x_n	y_n
0	1.0	2.0
1	1.1	2.104545
2	1.2	2.216667
3	1.3	2.334615
4	1.4	2.457143
5	1.5	2.583333

Exercise 187

- **Exercise 2.** Obtain a numerical solution of the differential equation: $\frac{1}{x}\frac{dy}{dx}+2y=1$ using the Runge-Kutta method in the range x = 0(0.2)1.0, given the initial conditions that x = 0 when y = 1. (Please use EXCEL to obtain the table.)

Exercise 187

- **Exercise 2. (cont.)**

x_n	k_1	k_2	k_3	k_4	y_n
0					1.000000
0.2	0.000000	-0.100000	-0.098000	-0.192160	0.980395
0.4	-0.192158	-0.276707	-0.271634	-0.340854	0.926071
0.6	-0.340857	-0.391986	-0.386873	-0.418436	0.848838
0.8	-0.418605	-0.429768	-0.428205	-0.421115	0.763649
1.0	-0.421838	-0.398637	-0.402813	-0.366172	0.683952

Homework Assignment 4

Deadline: 20 May 2008

(Firm Real-Time)