Chapter 48: Linear First Order Differential Equations

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Outline

- Introduction
- Procedure to Solve Differential Equations of the Form $\frac{dy}{dx} + Py = Q$
- Worked Problems on Linear First Order Differential Equations
- Further Worked Problems on Linear First Order Differential Equations

Introduction (1/5)

- An equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only is called a linear differential equation since y and its derivatives are of the first degree.

1. The solution of $\frac{dy}{dx} + Py = Q$ is obtained by multiplying throughout by what is termed an integrating factor.

Introduction (2/5)

2. Multiplying $\frac{dy}{dx} + Py = Q$ by say R, a function of x only, gives: $R\frac{dy}{dx} + RPy = RQ$ (1)

3. The differential coefficient of a product Ry is obtained using the product rule,
   i.e., $\frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$,
   which is the same as the left hand side of equation (1), when R is chosen such that

$$RP = \frac{dR}{dx}$$

Introduction (3/5)

4. If $\dfrac{dR}{dx} = RP$, then separating the variables gives $\dfrac{dR}{R} = P\,dx$.

   Integrating both sides gives:

   $$\int \frac{dR}{R} = \int P\,dx \text{ i.e., } \ln R = \int P\,dx + c$$

   from which,

   $$R = e^{\int P\,dx + c} = e^{\int P\,dx} e^{c}$$

   i.e., $R = Ae^{\int P\,dx}$, where $A = e^{c}$ = a constant.

Introduction (4/5)

5. Substituting $R = Ae^{\int P\,dx}$ in equation (1) gives:

   $$Ae^{\int P\,dx}\left(\frac{dy}{dx}\right) + Ae^{\int P\,dx} Py = Ae^{\int P\,dx} Q$$

   i.e., $$e^{\int P\,dx}\left(\frac{dy}{dx}\right) + e^{\int P\,dx} Py = e^{\int P\,dx} Q \qquad (2)$$

6. The left hand side of equation (2) is $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right)$ which may be checked by differentiating $ye^{\int P\,dx}$ with respect to x, using the product rule.

Introduction (5/5)

7. From equation (2), $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right) = e^{\int P\,dx} Q$

   Integrating both sides gives:

   $$ye^{\int P\,dx} = \int e^{\int P\,dx} Q\,dx \qquad (3)$$

8. $e^{\int P\,dx}$ is the integrating factor.

Procedure

- Rearrange the differential equation into the form $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x.
- Determine $\int P\,dx$.
- Determine the integration factor $e^{\int P\,dx}$.
- Substitute $e^{\int P\,dx}$ into equation (3).
- Integrate the right hand side of equation (3) to give the general solution of the differential equation. Given boundary conditions, the particular solution may be determined.

Worked Problems

- **Problem 1.** Solve $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+4y=2$ given the boundary conditions $x = 0$ when $y = 4$.

  $[y=\frac{1}{2}\left(1+7e^{-2x^2}\right)]$

- **Problem 2.** Show that the solution of the equation $\frac{\mathrm{d}y}{\mathrm{d}x}+1=-\frac{y}{x}$ is given by $y=\frac{3-x^2}{2x}$, given $x = 1$ when $y = 1$.

Worked Problems & Exercise 183

- **Problem 3.** Determine the particular solution of $\frac{\mathrm{d}y}{\mathrm{d}x}-x+y=0$, given that $x = 0$ when $y = 2$.

  $[y = x - 1 + 3e^{-x}]$

Solve the following differential equations.

- **Exercise 1.** $x\frac{\mathrm{d}y}{\mathrm{d}x}=3-y$

  $[y = 3 + c/x]$

Exercise 183

Solve the following differential equations.

- **Exercise 4.** $x\left(\frac{\mathrm{d}y}{\mathrm{d}x}+1\right)=x^3-2y$, given $x = 1$ when $y = 3$

  $[y = x^3/5 - x/3 + 47/15x^2]$

- **Exercise 6.** $\frac{\mathrm{d}y}{\mathrm{d}x}+x=2y$

  $[y = x/2 + 1/4 + ce^{2x}]$

Further Worked Problems (1/2)

- **Problem 4.** Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}\theta}=\sec\theta+y\tan\theta$ given the boundary conditions $y = 1$ when $\theta = 0$.

  $[y=(\theta+1)\sec\theta]$

Further Worked Problems (2/2)

- **Problem 5.**
  (a) Find the general solution of the equation

$$(x-2)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3(x-1)}{(x+1)}y=1$$

  (b) Given the boundary conditions that $y = 5$ when $x = -1$, find the particular solution of the equation given in (a).

  [(a) $y(x + 1)^2(x - 2) = (x + 1)^3/3 + c$
  (b) $y = (x + 1)/3(x - 2)$]

Exercise 184

- **Exercise 2.** Solve the differential equation:
  $t\frac{\mathrm{d}\theta}{\mathrm{d}t}+\sec t(t\sin t+\cos t)\theta=\sec t$, given $t = \pi$ when $\theta = 1$.

  [$\theta=\frac{1}{t}(\sin t-\pi\cos t)$]

Exercise 184

- **Exercise 6.** The equation $\frac{\mathrm{d}v}{\mathrm{d}t}=-(av+bt)$, where a and b are constants, represents an equation of motion when a particle moves in a resisting medium. Solve the equation for v given that $v = u$ when $t = 0$.

  [$v = b/a^2 - bt/a + (u - b/a^2)e^{-at}$]

Exercise 184

- **Exercise 9.** The equation of motion of a train is given by: $m\frac{\mathrm{d}v}{\mathrm{d}t}=mk(1-e^{-t})-mcv$, where v is the speed, t is time and m, k, and c are constants. Determine the speed, v, given $v = 0$ at $t = 0$.

  [$v=k\left\{\frac{1}{c}-\frac{e^{-t}}{c-1}+\frac{e^{-ct}}{c(c-1)}\right\}$]