Chapter 46: Solution of First Order Differential Equations by Separation of Variables

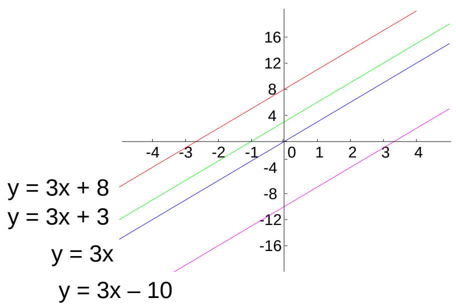
謝仁偉 助理教授
jenwei@mail.ntust.edu.tw
國立台灣科技大學 資訊工程系
2008 Spring

Outline

- Family of Curves
- Differential Equations
- The Solution of Equations of the Form $\frac{dy}{dx} = f(x)$
- The Solution of Equations of the Form $\frac{dy}{dx} = f(y)$
- The Solution of Equations of the Form $\frac{dy}{dx} = f(x) \cdot f(y)$

Family of Curves

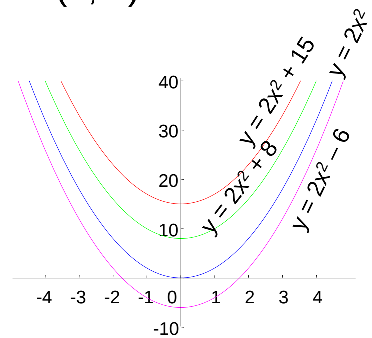
- Integrating both sides of the derivative $\frac{dy}{dx} = 3$ with respect to x gives $y = \int 3\,dx$, i.e., $y = 3x + c$, where c is an arbitrary constant.
- $y = 3x + c$ represents a family of curves, each of the curves in the family depending on the value of c.



Problems

- **Problem 1.** Sketch the family of curves given by the equation $dy/dx = 4x$ and determine the equation of one of these curves which passes through the point (2, 3).

  $[y = 2x^2 - 5]$



Differential Equations (1/4)

- A differential equation is one that contains differential coefficients.
- Differential equations are classified according to the highest derivative which occurs in them.

$$\frac{dy}{dx} = 7x$$ (first order differential equation)

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$$ (second order differential equation)

Differential Equations (2/4)

- The degree of a differential equation is that of the highest power of the highest differential which the equation contains after simplification.

$$\left(\frac{d^2x}{dt^2}\right)^3 + 2\left(\frac{dx}{dt}\right)^5 = 7$$ (second order differential equation of degree three)

- A solution to a differential equation which contains one or more arbitrary constants of integration is called the general solution of the differential equation.

Differential Equations (3/4)

- When additional information (boundary conditions) is given so that constants may be calculated, the particular solution of the differential equation is obtained.
- Equations which can be written in the form

$$\frac{dy}{dx} = f(x), \frac{dy}{dx} = f(y), \text{and } \frac{dy}{dx} = f(x) \cdot f(y)$$

can all be solved by integration.

Differential Equations (4/4)

$$\frac{dy}{dx} = f(x), \frac{dy}{dx} = f(y), \text{and } \frac{dy}{dx} = f(x) \cdot f(y)$$

- In each case, it is possible to separate the y's to one side of the equation and the x's to the other.
- Solving such equations is therefore known as solution by separation of variables.

The Solution of Equations of the Form $dy/dx = f(x)$

- A differential equation of the form $\frac{dy}{dx} = f(x)$ is solved by direct integration, i.e.,

$$y = \int f(x)\,dx$$

- **Problem 4.** Solve the equation $2x\left(x - \frac{dy}{dx}\right) = 5$, given $y = 2$ when $x = 1$.

  [$y = (x^2 - 5\ln x + 3)/2$]

Exercise 178

Solve the differential equations:

- **Exercise 2.** $2x\frac{dy}{dx} = 3 - x^3$

  [$y = \frac{3}{2}\ln x - \frac{x^3}{6} + c$]

- **Exercise 5.** $\frac{1}{e^x} + 2 = x - 3\frac{dy}{dx}$, given $y = 1$ when $x = 0$.

  [$y = \frac{1}{6}(x^2 - 4x + \frac{2}{e^x} + 4)$]

Exercise 178

- **Exercise 8.** An object is thrown vertically upwards with an initial velocity, u, of 20 m/s. The motion of the object follows the differential equation $ds/dt = u - gt$, where s is the height of the object in meters at time t seconds and g = 9.8 m/s^2. Determine the height of the object after 3 seconds if $s = 0$ when $t = 0$.

  [15.9m]

The Solution of Equations of the Form $dy/dx = f(y)$

- A differential equation of the form $\frac{dy}{dx} = f(y)$ is initially rearranged to give $dx = \frac{dy}{f(y)}$ and then the solution is obtained by direct integration, i.e.,

$$\int dx = \int \frac{dy}{f(y)}$$

Problems

- **Problem 7.** Determine the particular solution of $(y^2-1)\frac{\mathrm{d}y}{\mathrm{d}x}=3y$ given that $y = 1$ when $x = 13/6$
  $[x=\frac{y^2}{6}-\frac{1}{3}\ln y+2]$

Problems

- **Problem 8.** (a) The variation of resistance, R ohms, of an aluminum conductor with temperature θ ℃ is given by $\frac{\mathrm{d}R}{\mathrm{d}\theta}=\alpha R$, where α is the temperature coefficient of resistance of aluminum. If $R = R_0$ when $\theta = 0$℃, solve the equation for R. (b) If $\alpha = 38 \times 10^{-4}$/℃, determine the resistance of an aluminum conductor at 50℃, correct to 3 significant figures, when its resistance at 0℃ is 24.0 Ω.
  [(a) $R=R_0e^{\alpha\theta}$ (b) 29.0 ohms]

Exercise 179

- **Exercise 2.** Solve the differential equation:
  $\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos^2 y$
  [tan y = 2x + c]

- **Exercise 4.** The current in an electric circuit is given by the equation
  $$Ri+L\frac{\mathrm{d}i}{\mathrm{d}t}=0,$$
  where L and R are constants. Show that $i=Ie^{\frac{-Rt}{L}}$, given that $i = I$ when $t = 0$.

Exercise 179

- **Exercise 5.** The velocity of a chemical reaction is given by $\frac{\mathrm{d}x}{\mathrm{d}t}=k(a-x)$, where x is the amount transferred in time t, k is a constant and a is the concentration at time $t = 0$ when $x = 0$. Solve the equation and determine x in terms of t.
  [$x = a(1 - e^{-kt})$]

The Solution of Equations of the Form $dy/dx = f(x) \cdot f(y)$

- A differential equation of the form $\frac{dy}{dx} = f(x) \cdot f(y)$, where $f(x)$ is a function of x only, and $f(y)$ is a function of y only, may be rearranged as $\frac{dy}{f(y)} = f(x)\,dx$, and then the solution is obtained by direct integration, i.e.,

$$\int \frac{dy}{f(y)} = \int f(x)\,dx$$

Problems

- **Problem 9.** Solve the equation $4xy\frac{dy}{dx} = y^2 - 1$
  $[2\ln(y^2 - 1) = \ln x + c$ or
  $\frac{(y^2-1)^2}{x} = e^c$ or
  $(y^2 - 1)^2 = Ax]$
- **Problem 10.** Determine the particular solution of $\frac{d\theta}{dt} = 2e^{3t-2\theta}$, given that t = 0 when θ = 0.
  $[3e^{2\theta} = 4e^{3t} - 1]$

Exercise 180

- **Exercise 3.** Solve the differential equation: $\frac{dy}{dx} = e^{2x-y}$, given $x = 0$ when $y = 0$.
  $[e^y = \frac{1}{2}e^{2x} + \frac{1}{2}]$
- **Exercise 6.** Solve $xy = (1 - x^2)\frac{dy}{dx}$ for y, given x = 0 when $y = 1$.
  $[y = \frac{1}{\sqrt{(1-x^2)}}]$

Exercise 180

- **Exercise 8.** The p.d., V, between the plates of a capacitor C charged by a steady voltage E through a resistor R is given by the equation

$$CR\frac{dV}{dt} + V = E$$

a) Solve the equation for V given that at $t = 0$, $V = 0$.

b) Calculate V, correct to 3 significant figures, when $E = 25$V, $C = 20 \times 10^{-6}$F, $R = 200 \times 10^3\,\Omega$ and $t = 3.0$s.

$[(a)\ V = E(1 - e^{-t/CR})\ (b)\ 13.2\ V]$