Chapter 9: Solving Equations by Iterative Methods

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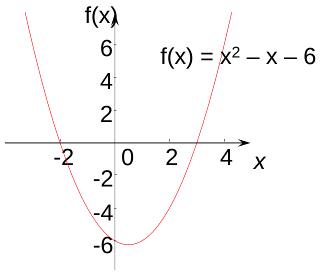
Outline

- Introduction to Iterative Methods
- The Bisection Method
- An Algebraic Method of Successive Approximations
- The Newton-Raphson Method

Introduction to Iterative Methods

- Each successive approximation method relies on a reasonably good first estimate of the value of a root being made.
- The value of the graph of f(x) = 0 changes sign for values of x just before and just after the value of a root.



Functional Notation:

$f(x) = x^2 - x - 6$
$f(2) = 2^2 - 2 - 6 = -4$
$f(4) = 4^2 - 4 - 6 = +6$

The Bisection Method

- If $f(x_1)$ and $f(x_2)$ have opposite signs, there is at least one root of the equation f(x) = 0 in the interval between x_1 and x_2 (provided f(x) is a continuous function).
- In the method of bisection the mid-point of the interval, i.e., $x_3 = (x_1 + x_2)/2$, is taken, and from the sign of $f(x_3)$, it can be deduced whether a root lies in the half interval to the left or right of x_3.

Problems

- **Problem 1.** Use the bisection method to find the positive root of the equation $5x^2 + 11x - 17 = 0$ correct to 3 significant figures.

f(0) = -17, f(1) = -1, f(2) = +25; (1 + 2)/2 = 1.5
f(1.5) = +10.75; (1 + 1.5)/2 = 1.25
f(1.25) = +4.5625; (1 + 1.25)/2 = 1.125
f(1.125) = +1.703125; (1 + 1.125)/2 = 1.0625
f(1.0625) = +0.33203125; (1 + 1.0625)/2 = 1.03125
f(1.03125) = -0.338867…; (1.03125 + 1.0625)/2 = 1.046875
f(1.046875) = -0.0046386…; (1.046875 + 1.0625)/2 = 1.0546875

[1.05]

An Algebraic Method of Successive Approximations (1/2)

- This method can be used to solve equations of the form: $a + bx + cx^2 + dx^3 + \ldots = 0$, where a, b, c, d, … are constants.

Procedure:

- *First approximation*
  a) Using a graphical or the functional notation method determine an approximate value of the root required, say x_1.
- *Second approximation*
  b) Let the true value of the root be $(x_1 + \delta_1)$.

An Algebraic Method of Successive Approximations (2/2)

c) Determine x_2, the approximate value of $(x_1 + \delta_1)$, by determining the value of $f(x_1 + \delta_1) = 0$, but neglecting terms containing products of δ_1.

- *Third approximation*

d) Let the true value of the root be $(x_2 + \delta_2)$.

e) Determine x_3, the approximate value of $(x_2 + \delta_2)$, by determining the value of $f(x_2 + \delta_2) = 0$, but neglecting terms containing products of δ_2.

f) The fourth and higher approximations are obtained in a similar way.

Using the techniques given in (b) to (f), it is possible to continue getting values nearer and nearer to the required root.

Problems

- **Problem 4.** Use an algebraic method of successive approximations to determine the value of the negative root of the quadratic equation: $4x^2 - 6x - 7 = 0$ correct to 3 significant figures. Check the value of the root by using the quadratic formula.

  [-0.771]

- **Problem 5.** Determine the value of the smallest positive root of the equation $3x^3 - 10x^2 + 4x + 7 = 0$, correct to 3 significant figures, using an algebraic method of successive approximations.

  [1.62]

Exercise 40

Use an algebraic method of successive approximation to solve the following equations to the accuracy stated.

- **Exercise 2.** $x^3 - 2x + 14 = 0$, correct to 3 decimal places.

  [-2.686]

- **Exercise 3.** $x^4 - 3x^3 + 7x - 5.5 = 0$, correct to 3 significant figures.

  [-1.53, 1.68]

The Newton-Raphson Method

- Newton-Raphson formula (Newton's method):
  If r_1 is the approximate value of a real root of the equation $f(x) = 0$, then a closer approximation to the root r_2 is given by:

  $$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

- Advantages of Newton's method:
  - Can be used for any type of mathematical equation
  - Usually easier to apply than the algebraic method

Problems & Exercise 41

- **Problem 7.** Taking the first approximation as 2, determine the root of the equation $x^2 - 3\sin x + 2\ln(x + 1) = 3.5$, correct to 3 significant figures, by using Newton's method.

  [2.01]

- **Exercise 5.** Use Newton's method to solve the equation $3\ln x + 4x = 5$, correct to 3 decimal places.

  [1.147]

Exercise 41

- **Exercise 9.** A Fourier analysis of the instantaneous value of a waveform can be represented by:

  $$y = \left(t + \frac{\pi}{4}\right) + \sin t + \frac{1}{8}\sin 3t$$

  Use Newton's method to determine the value of t near to 0.04, correct to 4 decimal places, when the amplitude, y, is 0.880.

  [0.0399]