# Chapter 7: The Binomial Series

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# Outline

- Pascal's Triangle
- The Binomial Series
- Worked Problems on the Binomial Series
- Further Worked Problems on the Binomial Series
- Practical Problems Involving the Binomial Theorem



# Pascal's Triangle (1/2)

- A binomial expression is one which contains two terms connected by a plus or minus sign.
- Expanding $(a + x)^n$ for integer values of $n$ form 0 to 6 gives the following results:

$(a + x)^0 =$ $1$
$(a + x)^1 = a + x$ $a + x$
$(a + x)^2 = (a + x)(a + x) =$ $a^2 + 2ax + x^2$
$(a + x)^3 = (a + x)^2(a + x) =$ $a^3 + 3a^2x + 3ax^2 + x^3$
$(a + x)^4 = (a + x)^3(a + x) =$ $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$
$(a + x)^5 = (a + x)^4(a + x) =$ $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$
$(a + x)^6 = (a + x)^5(a + x) =$ $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$



# Pascal's Triangle (2/2)

Pascal's Triangle

$(a + x)^0 = (a + x)^0$ 1
$(a + x)^1 = (a + x)^1$ 1 1 $a + x$
$(a + x)^2 = (a + x)(a + x) =$ 1 2 1 $a^2 + 2ax + x^2$
$(a + x)^3 = (a + x)^2(a + x) =$ 1 3 3 1 $a^3 + 3a^2x + 3ax^2 + x^3$
$(a + x)^4 = (a + x)^3(a + x) =$ 1 4 6 4 1 $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$
$(a + x)^5 = (a + x)^4(a + x) =$ 1 5 10 10 5 1 $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$
$(a + x)^6 = (a + x)^5(a + x) =$ 1 6 15 20 15 6 1 $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$

- 'a' decreases in power.
- 'x' increases in power.
- The coefficients are symmetrical.
- A coefficient of a term may be obtained by adding the two adjacent coefficients immediately above in the previous row.
- Applicable for $n < 8$.

# Problems

- **Problem 1.** Use the Pascal's triangle method to determine the expansion of $(a + x)^7$.
  $[(a + x)^7 = a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7]$

- **Problem 2.** Determine, using Pascal's triangle method, the expansion of $(2p – 3q)^5$.
  $[(2p - 3q)^5 = 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5]$



# Exercise 32

- **Exercise 2.** Expand $(2a + 3b)^5$ using Pascal's triangle.
  $[32a^5 + 240a^4b + 720a^3b^2 + 1080a^2b^3 + 810ab^4 + 243b^5]$



# The Binomial Series (1/2)

- The binomial series or binomial theorem is a formula for raising a binomial expression to any power without lengthy multiplication.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots$$

- The $r$'th term of the expansion of $(a + x)^n$ is

$$\frac{n(n-1)(n-2)\ldots \text{ to } (r-1) \text{ terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}$$



# The Binomial Series (2/2)

- If $a = 1$ in the binomial expansion of $(a + x)^n$ then:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

  which is valid for $-1 < x < 1$.

- When $x$ is small compared with 1 then:

$$(1+x)^n \approx 1 + nx$$

# Worked Problems on the Binomial Series

- **Problem 4.** Expand $(c - 1/c)^5$ using the binomial series.

  $[c^5 - 5c^3 + 10c - 10/c + 5/c^3 - 1/c^5]$
- **Problem 8.** Evaluate $(0.97)^6$ correct to 4 significant figures using the binomial expansion.

  [0.8330]
- **Problem 9.** Determine the value of $(3.039)^4$ correct to 6 significant figures using the binomial theorem.

  [85.2948]



# Exercise 33

- **Exercise 3.** Use the binomial theorem to expand $(2x - 3y)^4$.

  $[16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4]$
- **Exercise 7.** Determine the middle term of $(2a - 5b)^8$.

  $[700000a^4b^4]$
- **Exercise 9.** Use the binomial theorem to determine, correct to 5 significant figures: (a) $(0.98)^7$ (b) $(2.01)^9$

  [(a) 0.86813 (b) 535.51]



# Further Worked Problems on the Binomial Series (1/2)

- **Problem 11.** (a) Expand $1/(4 - x)^2$ in ascending power of x as far as the term in $x^3$, using the binomial theorem. (b) What are the limits of x for which the expansion in (a) is true?

  $[(a)\ (1 + x/2 + 3x^2/16 + x^3/16 + \ldots)/16$
  $(b)\ -4 < x < 4]$
- **Problem 12.** Use the binomial theorem to expand $\sqrt{4+x}$ in ascending powers of x to four terms. Give the limits of x for which the expansion is valid.

  $[2 + x/4 - x^2/64 + x^3/512 - \ldots, -4 < x < 4]$



# Further Worked Problems on the Binomial Series (2/2)

- **Problem 15.** Express $\dfrac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}}$ as a power series as far as the term in $x^2$. State the range of values of x for which the series is convergent.

  $[1 + 2x + 5x^2/2, -1/3 < x < 1/3]$

# Exercise 34

In problems 3 and 5 expand in ascending powers of x as far as the term in $x^3$, using the binomial theorem. State in each case the limits of x for which the series is valid.

- **Exercise 3.** $1/(2 + x)^3$

  [$1/8 - 3x/16 + 3x^2/16 - 5x^3/32 + \ldots$, $|x| < 2$]
- **Exercise 5.** $1/\sqrt{1+3x}$

  [$(1 - 3x/2 + 27x^2/8 - 135x^3/16 + \ldots)$, $|x| < 1/3$]



# Exercise 34

- **Exercise 7.** When x is very small show that:

  $$(1-2x)/(1-3x)^4 \approx 1+10x$$

- **Exercise 8.** If x is very small such that $x^2$ and higher powers may be neglected, determine the power series for $\dfrac{\sqrt{x+4}\sqrt[3]{8-x}}{\sqrt[5]{(1+x)^3}}$

  [$4 - 31x/15$]



# Practical Problems Involving the Binomial Theorem

- **Problem 16.** The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in (a) its volume and (b) its curved surface area, (neglecting the products of small quantities).

  [(a) reduced by approximately 6% (b) reduced by approximately 2%]



# Exercise 35

- **Exercise 6.** The electric field strength $H$ due to a magnet of length $2l$ and moment $M$ at a point on its axis distance $x$ from the centre is given by

  $$H = \frac{M}{2l}\left\{\frac{1}{(x-l)^2} - \frac{1}{(x+l)^2}\right\}$$

  Show that if $l$ is very small compared with $x$, then $H \approx \dfrac{2M}{x^3}$.

# Exercise 35

- **Exercise 9.** In a series electrical circuit containing inductance $L$ and capacitance $C$ the resonant frequency is given by $f_r = \frac{1}{2\pi\sqrt{LC}}$. If the values of $L$ and $C$ used in the calculation are 2.6% too large and 0.8% too small respectively, determine the approximate percentage error in the frequency.

  [0.9% too small]



# Exercise 35

- **Exercise 11.** A magnetic pole, distance $x$ from the plane of a coil of radius $r$, and on the axis of the coil, is subject to a force $F$ when a current flows in the coil. The force is given by: $F = \frac{kx}{\sqrt{(r^2 + x^2)^5}}$, where $k$ is a constant. Use the binomial theorem to show that when $x$ is small compared to $r$, then

  $$F \approx \frac{kx}{r^5} - \frac{5kx^3}{2r^7}$$