Chapter 5: Hyperbolic Functions

謝仁偉 助理教授
jenwei@mail.ntust.edu.tw
國立台灣科技大學 資訊工程系
2008 Spring

Outline

- Introduction to Hyperbolic Functions
- Graphs of Hyperbolic Functions
- Hyperbolic Identities
- Solving Equations Involving Hyperbolic Functions
- Series Expansions for cosh x and sinh x

Introduction to Hyperbolic Functions

- Hyperbolic sine of x,

$$\sinh x = \text{sh } x = \frac{e^x - e^{-x}}{2}$$

- Hyperbolic cosine of x,

$$\cosh x = \text{ch } x = \frac{e^x + e^{-x}}{2}$$

- Hyperbolic tangent of x,

$$\tanh x = \text{th } x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- Hyperbolic cotangent of x,

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

- Hyperbolic secant of x,

$$\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

- Hyperbolic cosecant of x,

$$\text{cosech } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Some Properties of Hyperbolic Functions

- sinh 0 = $(e^0 - e^{-0})/2 = (1 - 1)/2 = 0$
- cosh 0 = $(e^0 + e^{-0})/2 = (1 + 1)/2 = 1$
- $f(x)$ is an odd function of x if $f(-x) = -f(x)$
  - **sinh x** and **tanh x** are both odd functions.
  - As also are **cosech x** and **coth x**.
- $f(x)$ is an even function of x if $f(-x) = f(x)$
  - **cosh x** and **sech x** are both even functions.

Problems

- **Problem 3.** Evaluate, correct to 4 significant figures, (a) th 0.52 (b) cosech 1.4 (c) sech 0.86 (d) coth 0.38

  [(a) 0.4777 (b) 0.5251 (c) 0.7178 (d) 2.757]

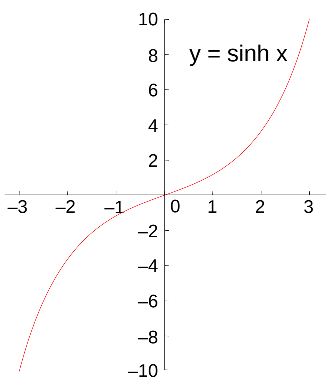
Exercise 24

- **Exercise 7.** A telegraph wire hangs so that its shape is described by y = 50 ch (x/50). Evaluate, correct to 4 significant figures, the value of y when x = 25.

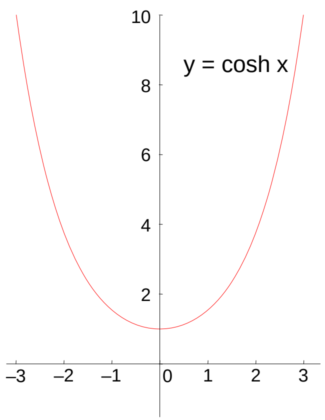
  [56.38]
- **Exercise 9.** $V^2 = 0.55L \tanh(6.3d/L)$ is a formula for velocity V of waves over the bottom of shallow water, where d is the depth and L is the wavelength. If d = 8.0 and L = 96, calculate the value of V.

  [5.042]

Graphs of Hyperbolic Functions (1/3)

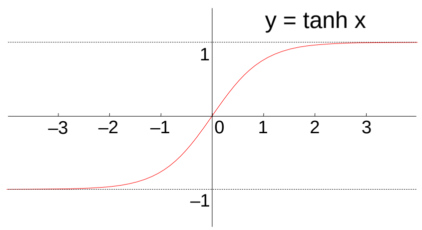


Since the graph is symmetrical about the origin, sinh x is an odd function.

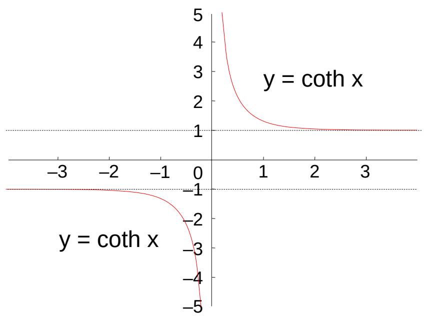


Since the graph is symmetrical about the y-axis, cosh x is an even function.

Graphs of Hyperbolic Functions (2/3)

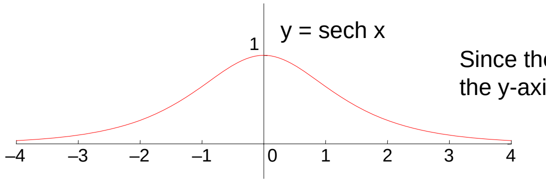


Since the graph is symmetrical about the origin, tanh x is an odd function.

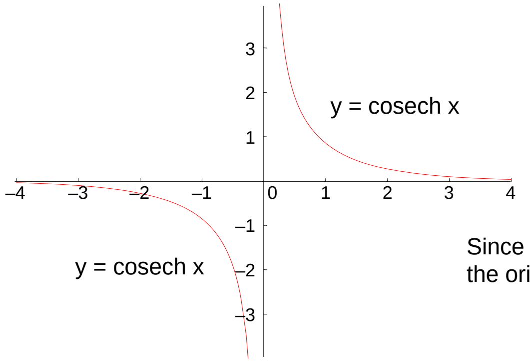


Since the graph is symmetrical about the origin, coth x is an odd function.

Graphs of Hyperbolic Functions (3/3)



Since the graph is symmetrical about the y-axis, sech x is an even function.



Since the graph is symmetrical about the origin, cosech x is an odd function.

Hyperbolic Identities

- For every trigonometric identity there is a corresponding hyperbolic identity.
- Hyperbolic identities may be proved by either
  - Replacing **sh x** by $(e^x - e^{-x})/2$ and **ch x** by $(e^x + e^{-x})/2$, or
  - By using Osborne's rule, which states: "*the six trigonometric ratios used in trigonometric identities relating general angles may be replaced by their corresponding hyperbolic functions, but the sign of any direct or implied product of two sines must be changed.*"

Trigonometric identity	Corresponding hyperbolic identity
$\cos^2 x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $\cot^2 x + 1 = \mathrm{cosec}^2 x$	$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$ $1 - \mathrm{th}^2 x = \mathrm{sech}^2 x$ $\mathrm{coth}^2 x - 1 = \mathrm{cosech}^2 x$
Compound angle formulae	
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\mathrm{sh}(A \pm B) = \mathrm{sh}\,A\,\mathrm{ch}\,B \pm \mathrm{ch}\,A\,\mathrm{sh}\,B$ $\mathrm{ch}(A \pm B) = \mathrm{ch}\,A\,\mathrm{ch}\,B \pm \mathrm{sh}\,A\,\mathrm{sh}\,B$ $\mathrm{th}(A \pm B) = \dfrac{\mathrm{th}\,A \pm \mathrm{th}\,B}{1 \pm \mathrm{th}\,A\,\mathrm{th}\,B}$
Double angles	
$\sin 2x = 2\sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1 = 1 - 2\sin^2 x$ $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\mathrm{sh}\,2x = 2\,\mathrm{sh}\,x\,\mathrm{ch}\,x$ $\mathrm{ch}\,2x = \mathrm{ch}^2 x + \mathrm{sh}^2 x$ $= 2\,\mathrm{ch}^2 x - 1 = 1 + 2\,\mathrm{sh}^2 x$ $\mathrm{th}\,2x = \dfrac{2\,\mathrm{th}\,x}{1 + \mathrm{th}^2 x}$

Problems (1/2)

- **Problem 6.** Prove the hyperbolic identities
  (a) $\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$ (b) $1 - \mathrm{th}^2 x = \mathrm{sech}^2 x$
  (c) $\mathrm{coth}^2 x - 1 = \mathrm{cosech}^2 x$.
- **Problem 7.** Prove, using Osborne's rule
  (a) $\mathrm{ch}\,2A = \mathrm{ch}^2 A + \mathrm{sh}^2 A$
  (b) $1 - \mathrm{th}^2 x = \mathrm{sech}^2 x$.
- **Problem 8.** Prove that $1 + 2\mathrm{sh}^2 x = \mathrm{ch}\,2x$.

Problems (2/2)

- **Problem 9.** Show that $\text{th}^2 x + \text{sech}^2 x = 1$.
- **Problem 11.** If $4e^x - 3e^{-x} \equiv P\text{sh}\, x + Q\text{ch}\, x$, determine the value of P and Q.

  [P = 7, Q = 1]

Exercise 25

Prove the given identities.

- **Exercise 1.** $\text{ch}\,(P - Q) \equiv \text{ch}\, P\, \text{ch}\, Q - \text{sh}\, P\, \text{sh}\, Q$
- **Exercise 2.** (a) $\coth x \equiv 2\text{cosech}\, 2x + \text{th}\, x$

  (b) $\text{ch}\, 2x - 1 \equiv 2\text{sh}^2 x$
- **Exercise 3.** $\text{sh}\, 2A \equiv 2\text{sh}\, A\, \text{ch}\, A$
- **Exercise 4.** $\dfrac{\text{sh}^2 x + \text{ch}^2 x - 1}{2\text{ch}^2 x \coth^2 x} \equiv \tanh^4 x$
- **Exercise 6.** If $5e^x - 4e^{-x} \equiv A\, \text{sh}\, x + B\, \text{ch}\, x$, find A and B.

  [A = 9, B = 1]

Solving Equations Involving Hyperbolic Functions

- Change $\text{sh}\, x$ to $(e^x - e^{-x})/2$ and $\text{ch}\, x$ to $(e^x + e^{-x})/2$
- Rearrange the equation into the form $pe^x + qe^{-x} + r = 0$, where p, q, and r are constants.
- Multiply each term by e^x, which produces an equation of the form $p(e^x)^2 + re^x + q = 0$
- Solve the quadratic equation $p(e^x)^2 + re^x + q = 0$ for e^x by factorizing or by using the quadratic formula.
- Given e^x = a constant, take Napierian logarithms of both side to give $x = \ln$ (constant)

Problems

- **Problem 12.** Solve the equation $\text{sh}\, x = 3$, correct to 4 significant figures.

  [1.818]
- **Problem 14.** A chain hangs in the form given by $y = 40\, \text{ch}\, (x/40)$. Determine, correct to 4 significant figures, (a) the value of y when x is 25 and (b) the value of x when $y = 54.30$.

  [(a) y = 48.07 (b) x = ±32.89]

Exercise 26

Solve the equations correct to 4 decimal places.

- **Exercise 3.** $3.5\text{sh}\, x + 2.5\text{ch}\, x = 0$
  [-0.8959]
- **Exercise 5.** $4\text{th}\, x - 1 = 0$
  [0.2554]
- **Exercise 6.** A chain hangs so that its shape is of the form $y = 56\ \text{ch}\ (x/56)$. Determine, correct to 4 significant figures, (a) the value of y when x is 35, and (b) the value of x when y is 62.35.
  [(a) 67.30 (b) ±26.42]

Series Expansions for cosh x and sinh x

- By definition,
  $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5!\ldots$
- Replacing x by –x gives:
  $e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! - x^5/5!\ldots$
- cosh $x = (e^x + e^{-x})/2 = 1 + x^2/2! + x^4/4! + \ldots$
- sinh $x = (e^x - e^{-x})/2 = x + x^3/3! + x^5/5!\ldots$

Problem and Exercise 27

- **Problem 17.** Determine the power series for $2\text{ch}(x/2) - \text{sh}2x$ as far as the term in x^5.
  [$2 - 2x + x^2/4 - 4x^3/3 + x^4/192 - 4x^5/15 + \ldots$]

- **Exercise 5.** Prove the given identity, the series being taken as far as the term in x^5 only.
  $2\ \text{sh}\ (x/2) - \text{ch}\ (x/2) \equiv -1 + x - x^2/8 + x^3/24 - x^4/384 + x^5/1920$

Homework Assignment 1

Deadline: 17 March 2008

(Firm Real-Time)