Chapter 4: Logarithms and Exponential Functions

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Outline

Introduction to Logarithms

- If $y = a^x$ then $x = \log_a y$
- Logarithms having a base of 10 are called common logarithms and $\log_{10}$ is usually abbreviated to lg.
- Logarithms having a base of e (a mathematical constant approximately equal to 2.7183) are called hyperbolic, Napierian, or natural logarithms, and $\log_e$ is usually abbreviated to ln.

Laws of Logarithms & Problems

- $\log(A \times B) = \log A + \log B$
- $\log(A / B) = \log A - \log B$
- $\lg A^n = n \log A$

- **Problem 1 & 2.** Evaluate (a) $\log_{16} 8$ (b) $\log_3(1/81)$
  [(a) $\log_{16} 8 = 3/4$, (b) $\log_3(1/81) = -4$]
- **Problem 3.** Solve the following equations: (a) $\lg x = 3$ (b) $\log_2 x = 3$ (c) $\log_5 x = -2$.
  [(a) $x = 1000$ (b) $x = 8$ (c) $x = 1/25$]

Problems

- **Problem 4.** Write log450 in terms of log2, log3, and log5 to any base.
  [log2 + 2log3 + 2log5]
- **Problem 5.** Write $\log\left(\frac{8\times\sqrt[4]{5}}{81}\right)$ in terms of log2, log3, and log5 to any base.
  [3log2 + (1/4)log5 - 4log3]
- **Problem 6.** Evaluate $\frac{\log 25-\log 125+\frac{1}{2}\log 625}{3\log 5}$
  [1/3]

Exercise 16

- **Exercise 17.** Write $\log(125\times\sqrt[4]{16}/\sqrt[4]{81^3})$ in terms of log2, log3, and log5 to any base.
  [log2 - 3log3 + 3log5]
- **Exercise 20.** Evaluate $(\frac{1}{2}\log 16-\frac{1}{3}\log 8)/\log 4$
  [1/2]
- **Exercise 21.** Solve the equation: $\log x^4 - \log x^3 = \log 5x - \log 2x$.
  [x = 5/2]

Indicial Equations & Problems

- The laws of logarithms may be used to solve certain equations involving powers – called indicial equations.
- **Problem 8.** Solve the equation $2^x = 3$, correct to 4 significant figures.
  [1.585]
- **Problem 9.** Solve the equation $2^{x+1} = 3^{2x-5}$ correct to 2 decimal places.
  [4.11]

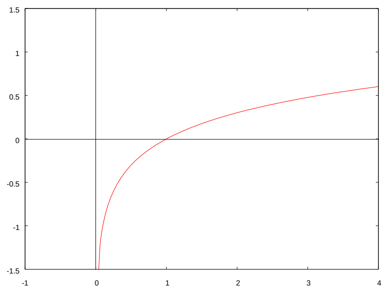
Exercise 17

Solve the following indicial equations for x, each correct to 4 significant figures.

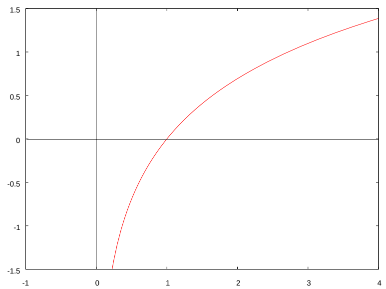
- **Exercise 1.** $3^x = 6.4$
  [1.690]
- **Exercise 3.** $2^{x-1} = 3^{2x-1}$
  [0.2696]
- **Exercise 4.** $x^{1.5} = 14.91$
  [6.058]

Graphs of Logarithmic Functions

$y = \log_{10}x$



$y = \log_e x$



- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a 0 \to -\infty$

The Exponential Function & Problems

- An exponential function is one which contains e^x
  - e: a constant called the exponent
  - $e \approx 2.7183$

- **Problem 11.** Use a calculator to determine the following, each correct to 4 significant figures: (a) $3.72e^{0.18}$ (b) $53.2e^{-1.4}$ (c) $5e^7/122$.
  [(a) 4.454 (b) 13.12 (c) 44.94]
- **Exercise 1.** Evaluate, correct to 5 significant figures: (a) $3.5e^{2.8}$ (b) $-6e^{-1.5}/5$ (c) $2.16e^{5.7}$
  [(a) 57.556 (b) -0.26776 (c) 645.55]

The Power Series for e^x

- The value of e^x is defined in terms of the following power series:

  $$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$$

- The value of $e^{0.05}$, correct to say 8 significant figures, is found by substituting x = 0.05 in the power series for e^x. Thus

$$e^{0.05} = 1 + 0.05 + \frac{(0.05)^2}{2!} + \frac{(0.05)^3}{3!} + \frac{(0.05)^4}{4!} + \frac{(0.05)^5}{5!} + \ldots$$

$$= 1 + 0.05 + 0.00125 + 0.000020833$$

$$+ 0.000000260 + 0.000000003$$

$$= 1.0512711$$

Problems

- **Problem 14.** Determine the value of $5e^{0.5}$, correct to 5 significant figures by using the power series for e^x.

$$e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{(2)(1)} + \frac{(0.5)^3}{(3)(2)(1)} + \frac{(0.5)^4}{(4)(3)(2)(1)}$$

$$+ \frac{(0.5)^5}{(5)(4)(3)(2)(1)} + \frac{(0.5)^6}{(6)(5)(4)(3)(2)(1)}$$

$$= 1 + 0.5 + 0.125 + 0.020833 + 0.0026042$$

$$+ 0.0002604 + 0.0000217$$

$$= 1.64872$$

$5e^{0.5} = 5(1.64872) = 8.2436$

Problems & Exercise 19

- **Problem 15.** Expand $e^x(x^2 - 1)$ as far as the term in x^5.
  [$-1 - x + x^2/2 + 5x^3/6 + 11x^4/24 + 19x^5/120$]
- **Exercise 3.** Expand $(1 - 2x)e^{2x}$ as far as the term in x^4.
  [$1 - 2x^2 - 8x^3/3 - 2x^4$]

Graphs of Exponential Functions

- **Problem 16.** Plot a graph of $y = 2e^{0.3x}$ over a range of $x = -2$ to $x = 3$. Hence determine the value of y when x = 2.2 and the value of x when y = 1.6

x	–3	–2	–1	0	1	2	3
$0.3x$	–0.9	–0.6	–0.3	0	0.3	0.6	0.9
$e^{0.3x}$	0.407	0.549	0.741	1.000	1.350	1.822	2.460
$2e^{0.3x}$	0.81	1.10	1.48	2.00	2.70	3.64	4.92

[From the graph, when x = 2.2, y = 3.87 and when y = 1.6, x = -0.74]

Using Gnuplot to Plot Graphs (1/5)

- Gnuplot Homepage: http://www.gnuplot.info/
- Command-line driven program for producing 2D and 3D plots
- Mathematical expression accepted by C, FORTRAN, Pascal, or BASIC may be plotted.
- Multiplication must be explicit, e.g., 5x are denoted by 5*x.
- Exponents are denoted by **.

* The content of the following 3 slides (inclusive) is extracted from the report of Ching-Yung Hung and Chuan-Yue Yang.

Using Gnuplot to Plot Graphs (2/5)

Supported Functions:

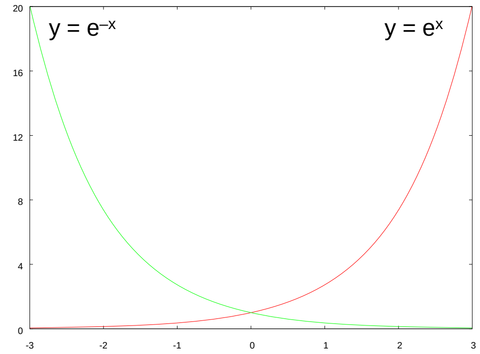
- Standard trig functions: sin, cos, tan
- Inverse trig functions: asin, acos, atan
- Hyperbolic trig functions: sinh, cosh, tanh
- Exponential function (base e): exp
- Logarithm functions: log (base e), log10
- Absolute value: abs
- Square root: sqrt

Using Gnuplot to Plot Graphs (3/5)

- 2D: plot <function>
  - plot x**3
- 3D: splot <function>
  - splot sin(x)+cos(y/5)

Using Gnuplot to Plot Graphs (4/5)

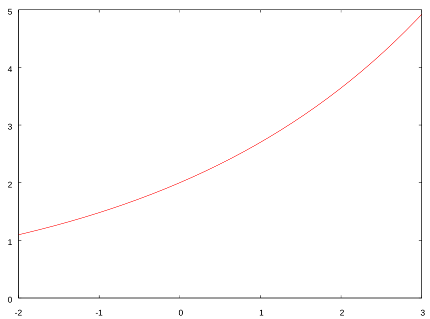
```
set xrange [-3:3]
set yrange [0:20]
set ytics 4
unset key
plot exp(x), exp(-x)
```



Using Gnuplot to Plot Graphs (5/5)

- **Problem 16.** Plot a graph of $y = 2e^{0.3x}$ over a range of x = –2 to x = 3. Hence determine the value of y when x = 2.2 and the value of x when y = 1.6

```
set xrange [-2:3]
set yrange [0:5]
plot 2*exp(0.3*x)
print 2*exp(0.3*2.2)
3.86958466880406
print log(0.8)/0.3
-0.743811837714032
```



Napierian Logarithms

- **Problem 21.** Solve the equation $7 = 4e^{-3x}$ to find x, correct to 4 significant figures.
  [x = ln(4/7)/3 = -0.1865]

- **Problem 22.** Given $20 = 60(1 - e^{-t/2})$ determine the value of t, correct to 3 significant figures.
  [t = 2ln(3/2) = 0.811]

Exercise 21

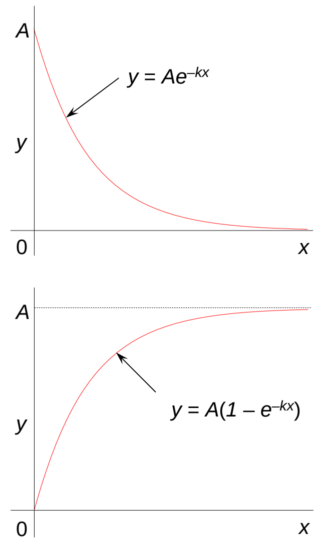
Solve the given equations, each correct to 4 significant figures.

- **Exercise 4.** $7.83 = 2.91e^{-1.7x}$

  [-0.5822]
- **Exercise 6.** 5.17 = ln(x/4.64)

  [816.2]

Laws of Growth and Decay



- Linear expansion

  $l = l_0 e^{\alpha\theta}$
- Change in electrical resistance with temperature

  $R_\theta = R_0 e^{\alpha\theta}$
- Newton's law of cooling

  $\theta = \theta_0 e^{-kt}$
- Biological growth

  $y = y_0 e^{kt}$
- Discharge of a capacitor

  $q = Qe^{-t/CR}$
- Growth of current in a capacitive circuit

  $i = I(1 - e^{-t/CR})$

… …

Problems

- **Problem 24.** The resistance R of an electrical conductor at temperature θ ℃ is given by $R = R_0 e^{\alpha\theta}$, where α is a constant and $R_0 = 5 \times 10^3$ ohms. Determine the value of α, correct to 4 significant figures, when R = 6 x 10^3 ohms and θ = 1500℃. Also, find the temperature, correct to the nearest degree, when the resistance R is 5.4 x 10^3 ohms.

  [α =1.215 x 10^{-4}, 633℃]

Exercise 22

- **Exercise 3.** The length l meters of a metal bar at temperature t℃ is given by $l = l_0 e^{\alpha t}$, where l_0 and α are constants. Determine (a) the value of α when l = 1.993m, l_0 = 1.894m and t = 250℃, and (b) the value of l_0 when l = 2.416, t = 310 ℃, and α = 1.682 x 10^{-4}.

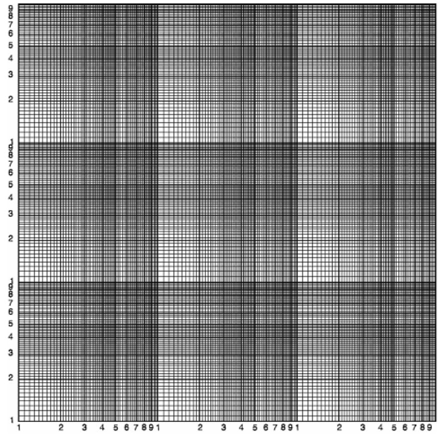
  [(a) 2.038 x 10^{-4} (b) 2.293m]

Exercise 22

- **Exercise 7.** The current i flowing in a capacitor at time t is given by: $i = 12.5(1 - e^{-t/CR})$ where resistance R is 30 kilohms and the capacitance C is 20 micro-farads. Determine:
  (a) the current flowing after 0.5 seconds, and
  (b) the time for the current to reach 10 amperes
  [(a) 7.07A (b) 0.966s]

Reduction of Exponential Laws to Linear Form (1/2)

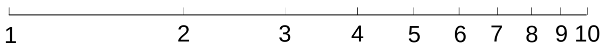
- Determination of law: The non-linear equation may be modified to the linear form, $y = mx + c$, so that the constants, and thus the law relating the variables can be determined.
- Log-log graph paper: The scale markings along the horizontal and vertical axes are proportional to the logarithms of the numbers.



http://www.engj.ulst.ac.uk/sidk/graph/graph.htm

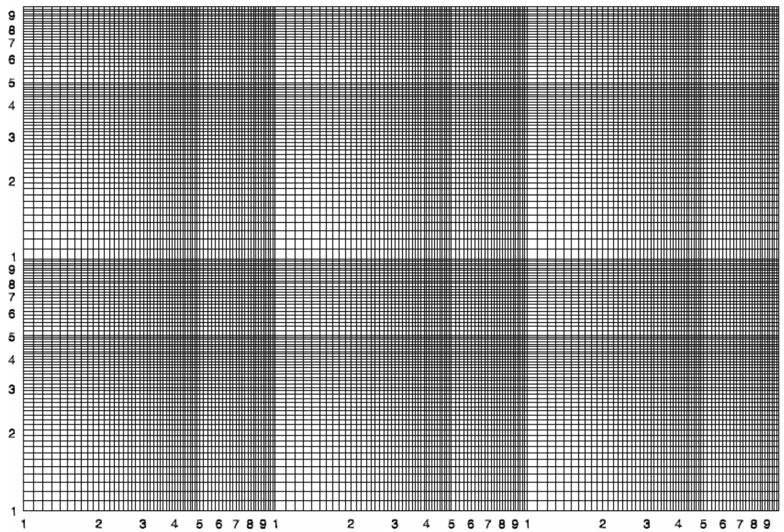
Reduction of Exponential Laws to Linear Form (2/2)

- Logarithmic scale:
  - The distance between, say 1 and 2, is proportional to lg2 – lg1, i.e., 0.3010 of the total distance from 1 to 10.
  - The distance between markings progressively decreases as the numbers increase from 1 to 10.



- Cycles: The number of times the pattern of markings is repeated on an axis.
  - e.g., "log 2 cycle x 3 cycle"
  - To depict a set of values from 0.4 to 161, on an axis of log-log graph paper, 4 cycles are required.

Log 2 Cycle x 3 Cycle



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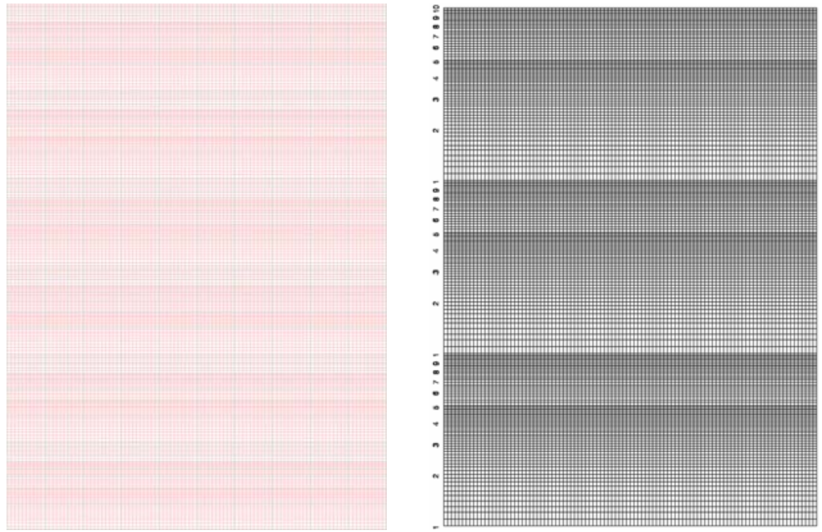
Graphs of the Form $y = ae^{kx}$

- Taking logarithms to a base of e of both sides of $y = ae^{kx}$ gives: $\ln y = kx + \ln a$ which compares with $Y = mX + c$
- By plotting $\ln y$ vertically against x horizontally, a straight line results, i.e., the equation $y = ae^{kx}$ is reduced to linear form.
- Log-linear graph paper: Graph paper has a linear horizontal scale and a logarithmic vertical scale.

"Log 3 Cycle x Linear" Graph Paper



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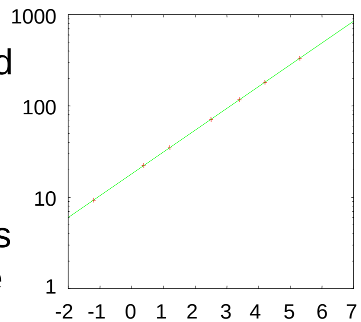
Problems

- **Problem 28.** The data given below is believed to be related by a law of the form $y = ae^{kx}$, where a and k are constants. Verify that the law is true and determine approximate values of a and k. Also determine the value of y when x is 3.8 and the value of x when y is 85.



[y = $18e^{0.55x}$, 146, 2.82]

x	-1.2	0.38	1.2	2.5	3.4	4.2	5.3
y	9.3	22.2	34.8	71.2	117	181	332

Exercise 23

- **Exercise 1.** Atmospheric pressure p is measured at varying altitudes h and the results are as shown below:

Altitude, h m	Pressure, p cm
500	73.39
1500	68.42
3000	61.60
5000	53.56
8000	43.41

Show that the quantities are related by the law $p = ae^{kh}$, where a and k are constants. Determine the values of a and k and state the law. Find also the atmospheric pressure at 10,000m.

[a = 76, k = -7×10^{-5}, 37.74 cm]