Theorem 5. *In a fixed-priority system where every job completes before the next job in the same task is released, a critical instant of any task T_i occurs when one of its job $J_{i,c}$ is released at the same time with a job in every higher-priority task, that is, $r_{i,c} = r_{k,l_k}$ for some l_k for every $k = 1, 2, ..., i-1$.*

Proof. We first show that the response time of the first job $J_{i,1}$ is the largest when the tasks in the subset $\mathbf{T}_i$ of i highest priority tasks are in phase, that is, when the condition stated in the theorem is met. To do so, we take as the time origin the minimum of all the phases of tasks in $\mathbf{T}_i$. It suffices for us to consider the case where the processor remains busy executing jobs with higher priorities than $J_{i,1}$ is released at ϕ_i.[1]

Let $W_{i,1}$ denote the response time of $J_{i,1}$. From the release time ϕ_k of the first job in T_k to the instant $\phi_i + W_{i,1}$ when the first job $J_{i,1}$ in T_i completes, at most $\lceil (W_{i,1} + \phi_i - \phi_k)/p_k \rceil$ job in T_k become ready for execution. Each of these jobs demands e_k units of processor time. Hence the total amount of processor time demanded by $J_{i,1}$ and all the jobs that must be completed before $J_{i,1}$ is given by

$$e_i + \sum_{k=1}^{i-1} \left\lceil \frac{W_{i,1} + \phi_i - \phi_k}{p_k} \right\rceil e_k$$

At time $W_{i,1} + \phi_i$ when $J_{i,1}$ completes, the supply of processor time becomes sufficient to meet this total demand for the processor time for first time since time 0. In other words, $W_{i,1}$ is equal to the smallest of all solutions of

$$W_{i,1} = e_i + \sum_{k=1}^{i-1} \left\lceil \frac{W_{i,1} + \phi_i - \phi_k}{p_k} \right\rceil e_k - \phi_i \tag{2}$$

if this equation has solutions in the range $(0, p_i]$. If the equation does not have a solution in this range, then $J_{i,1}$ cannot complete in time and misses its deadline.

To see how $W_{i,1}$ depends on the phase ϕ_k of each higher-priority task, we note that the expression in the right-hand side of this equation is a staircase function of $W_{i,1}$. It lies above the 45° straight line $y(W_{i,1}) = W_{i,1}$ until it intersects the straight line; the value of $W_{i,1}$ at this intersection is the solution of Equation 2. From Equation 2, it is clear that the staircase function has the largest value, and hence its intersection with the 45° straight line has the largest value, when ϕ_k is 0 for every $k = 1, 2, ..., i-1$. In other words, the job $J_{i,1}$ has the largest response time when all the higher-priority tasks are in phase.

We now let ϕ_k be 0 for all $k < i$. The time instant when the processor first completes all the ready jobs in higher-priority tasks released since time 0

[1] If during some intervals before ϕ_i the processor idles or executes lower priority jobs, we can ignore the segment of the schedule before the end of the latest of such intervals, take this time instant as the time origin, and call the first job in every higher-priority task released after this instant the first job of the task.

is independent when $J_{i,1}$ is released, and this is the first instant when $J_{i,1}$ can begin execution. Therefore, the sooner $J_{i,1}$ is released, the longer it must wait to start execution and the larger its response time. This observation leads us to conclude that the response time $W_{i,1}$ of $J_{i,1}$ has the largest possible value when ϕ_i is also equal to zero.

To show that an arbitrary job $J_{i,c}$ in T_i has the maximum possible response time whenever it is released at the same time with a job in every higher-priority task, we let the release time of this job be ϕ'_i and the release time of the job in T_k that is current at time ϕ'_i be ϕ'_k. Because every job completes in the period in which it is released, there is no need for us to consider the jobs in each task T_k that are released before ϕ'_k. Hence we can take the minimum among the release time ϕ'_k, for $k = 1, 2, ..., i$, of the current jobs as the time origin and consider the segment of the schedule starting from this new origin independently from the earlier segment. The above argument can be repeated and allows us to reach the conclusion that the response time of $J_{i,c}$ has the maximum value when the release times of all the current jobs in $\mathbf{T}_i$ are the same.

It follows from the above arguments that the maximum possible response time W_i of all jobs in T_i is, therefore, equal to the smallest value of t that satisfies the equation

$$t = e_i + \sum_{k=1}^{i-1} \left\lceil \frac{t}{p_k} \right\rceil e_k \tag{3}$$

It is easy to see that if Equation 2 does not have a solution equal to or less than D_i, neither does this equation.

□