Chapter 1: Fundamental Principles of Counting

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Outline

- ***The Rules of Sum and Product***
- Permutations
- Combinations: The Binomial Theorem
- Combinations with Repetition

The Rules of Sum (1/2)

- The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

The Rules of Sum (2/2)

- **Example 1.1:** A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among 40 + 50 = 90 textbooks in order to learn more about one or the other of these two subjects.

The Rules of Product (1/2)

- The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

The Rules of Product (2/2)

- **Example 1.6:** Considering the manufacture of license plates consisting of 2 letters followed by 4 digits.
  a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3{,}276{,}000$ different possible plates.
  b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6{,}760{,}000$ different license plates are possible.
  c) If repetitions are allowed, as in part (b), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

The Rules of Sum and Product

- **Example 1.8:** At the AWL corporation Mrs. Foster operates the Quick Snake Coffee Shop. The menu at her shop is limited: 6 kinds of muffins, 8 kinds of sandwiches, and 5 beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch – either a muffin and a hot beverage or a sandwich and a cold beverage. How many ways in which Carl can purchase Ms. Dodd's lunch?

  $6 \times 2 + 8 \times 3 = 12 + 24 = 36$

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Permutations (1/2)

- **Example 1.9:** In a class of 10 students, 5 are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

  $10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$

- **Definition 1.1:** For an integer $n \geq 0$, n factorial (denoted $n!$) is defined by

$$0! = 1,$$

$$n! = (n)(n-1)(n-2)\ldots(3)(2)(1), \quad \text{for } n \geq 1.$$

Permutations (2/2)

- **Definition 1.2:** Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with Indistinguishable Objects (1/2)

- **Example 1.12:** Consider the arrangements of all 9 letters in DATABASES.

  (2!)(3!)(Number of arrangements of the letters in DATABASES) = (Number of permutations of the symbols D, A_1, T, A_2, B, A_3, S_1, E, S_2)

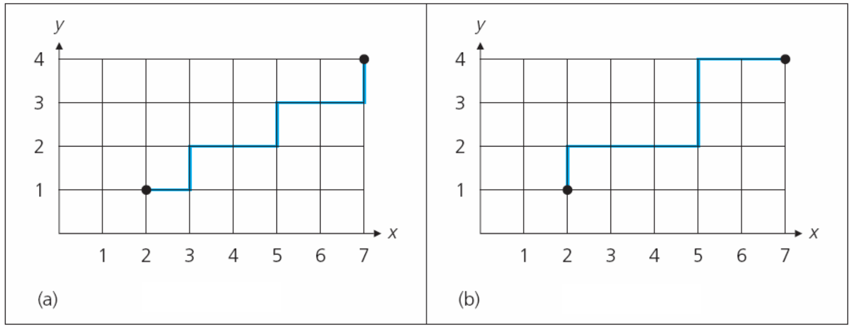
  9!/(2!3!) = 30,240

Permutations with Indistinguishable Objects (2/2)

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, ..., and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \ldots + n_r = n$, then there are $\frac{n!}{n_1! n_2! \ldots n_r!}$ (linear) arrangements of the given n objects.

Examples (1/2)

- **Example 1.14:** Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). ➡ 8!/(5!3!) = 56

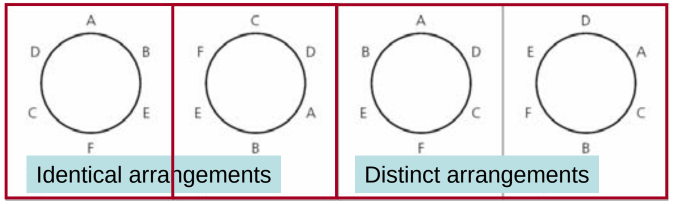


Examples (2/2)

- **Example 1.15:** (combinatorial proof) Prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer.
(Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$.)

Nonlinear Arrangement (1/2)

- **Example 1.16:** If 6 people, designated as A, B, …, F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

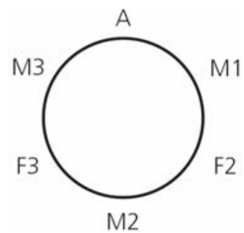


There are 6!/6 = 5! = 120 arrangements.

Nonlinear Arrangement (2/2)

- **Example 1.17:** Suppose now the 6 people of Example 1.16 are 3 married couples and that A, B, and C are the females. We want to arrange the 6 people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

$3 \times 2 \times 2 \times 1 \times 1 = 12$ ways



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- Combinations with Repetition

Combinations: The Binomial Theorem

- If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

Combination Examples (1/2)

- **Example 1.19:** Lynn and Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall, one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. How many ways can Lynn and Patti select for their ticket?

  C(49, 5)C(42, 1) = 80,089,128 ways

Combination Examples (2/2)

- **Example 1.20:**
  a) A student taking a history examination is directed to answer any 7 of 10 essay questions.
  b) The student must answer 3 questions from the first 5 and 4 questions from the last 5.
  c) The student must answer 7 of 10 questions where at least 3 are selected from the first 5.

  a) C(10, 7) = 120 ways

  b) C(5, 3)C(5,4) = 10 $\times$ 5 = 50 ways

  c) C(5, 3)C(5, 4) + C(5, 4)C(5, 3) + C(5, 5)C(5, 2) = 110 ways

Arrangements and Combinations (1/2)

- **Example 1.23:** The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!2!2!2!1!\,1!} = 831{,}600.$$

How many of these arrangement have no adjacent A's?

$$\frac{8!}{2!2!2!1!1!} \times C(9,3) = 5040 \times 84$$
$$= 423{,}360 \text{ arrangements}$$

Arrangements and Combinations (2/2)

- **Example 1.24:** Consider strings made up from symbols 0, 1, and 2. Suppose $x = x_1x_2x_3 \ldots x_n$ is one of strings of length n. We define the weight of x, denoted $\mathrm{wt}(x)$, by $\mathrm{wt}(x) = x_1 + x_2 + x_3 + \ldots + x_n$. Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

$$2^{10} + C(10,2)2^8 + C(10,4)2^6 + C(10,6)2^4 + C(10,8)2^2$$
$$+ C(10,10) = \sum_{n=0}^{5} C(10,2n)2^{10-2n}$$

Overcounting (1/2)

- **Example 1.25:**
  a) Suppose that Ellen draws 5 cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs?
  C(39, 5)
  b) Now suppose we want to count the number of Ellen's 5-card selections that contain at least one club.
  C(52, 5) – C(39, 5) = 2,023,303 vs.
  C(13, 1)C(51, 4) = 3,248,700
  Overcounting!

Overcounting (2/2)

- **Example 1.25 (cont.):**
  Another way to arrive at the answer:

$$C(13,1)C(39,4) + C(13,2)C(39,3) + C(13,3)C(39,2)$$
$$+ C(13,4)C(39,1) + C(13,5)C(39,0)$$
$$= \sum_{i=1}^{5} C(13,i)C(39,5-i)$$
$$= (13)(82{,}251) + (78)(9139) + (286)(741) + (715)(39)$$
$$+ (1287)(1) = 2{,}023{,}203.$$

The Binomial Theorem (1/3)

- **Theorem 1.1** The Binomial Theorem. If x and y are variables and n is a positive integer, then

$$(x+y)^n = C(n,0)x^0y^n + C(n,1)x^1y^{n-1} + \ldots + C(n,n)x^ny^0$$
$$= \sum_{k=0}^{n} C(n,k)x^ky^{n-k}.$$

  - There are $C(n, k)$ different ways to select k x's and $n-k$ y's from the n available factors.
  - $C(n, k)$ is often referred to as a binomial coefficient

The Binomial Theorem (2/3)

- **Example 1.26:**
  a) What is the coefficient of x^5y^2 in the expansion of $(x + y)^7$?
  b) What is the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$?

  a) C(7, 5) = 21

  b) C(7, 5)(2)^5(–3)^2 = 6048

The Binomial Theorem (3/3)

- **Corollary 1.1** For each integer $n > 0$,
  a) $C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n$, and
  b) $C(n, 0) - C(n, 1) + \ldots + (-1)^nC(n, n) = 0$

- **Proof.**
  a) Set $x = y = 1$
  b) Set $x = -1$, $y = 1$

The Multinomial Theorem (1/3)

- **Theorem 1.2** The Multinomial Theorem. For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\ldots x_t^{n_t}$ in the expansions of $(x_1 + x_2 + \ldots + x_t)^n$ is

$$\frac{n!}{n_1!n_2!n_3!\ldots n_t!}$$

  where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

The Multinomial Theorem (2/3)

- **Proof of Theorem 1.2:**
  - The coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, …

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} = \frac{n!}{n_1!n_2!\cdots n_t!} = \binom{n}{n_1, n_2, \cdots, n_t}$$

a multinomial coefficient

The Multinomial Theorem (3/3)

- **Example 1.27:**

  a) What is the coefficient of x^3z^4 in the expansion of $(x + y + z)^7$?
  b) What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$?

a) $$\binom{7}{3, 0, 4} = \frac{7!}{3!\,0!\,4!} = 35$$

b) $$\binom{16}{2, 3, 2, 5, 4}(1)^2(2)^3(-3)^2(2)^5(5)^4$$

$$= 435{,}891{,}456{,}000{,}000$$

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Combinations with Repetition (1/2)

- **Example 1.28:** 7 high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or … many different purchases … viewpoint of the

The 7 x's (one for each freshman) correspond to the size of the selection and the 3 bars are needed to separate the 3 + 1 = 4 possible food items that can be chosen.

(a)	(b)
1. c, c, h, h, t, t, f	1. x x | x x | x x | x
2. c, c, c, c, h, t, f	2. x x x x | x | x | x
3. c, c, c, c, c, c, f	3. x x x x x x | | | x
4. h, t, t, f, f, f, f	4. | x | x x | x x x x
5. t, t, t, t, t, f, f	5. | | x x x x x | x x
6. t, t, t, t, t, t, t	6. | | x x x x x x x |
7. f, f, f, f, f, f, f	7. | | | x x x x x x x

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

Combinations with Repetition (2/2)

When we wish to select, with repetition, r of n distinct objects, we find that we are considering all arrangements of r x's and $n - 1$ |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Consequently, the number of combinations of n objects taken r at a time, with repetition, is $C(n + r - 1, r)$.

Example

- **Example 1.29:** A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, how many ways can we select a dozen donuts?

  C(20 + 12 – 1, 12) = C(31, 12) = 141,120,525 ways.

Combination with Repetition + Rule of Product

- **Example 1.31:** In how many ways can we distribute 7 bananas and 6 oranges among 4 children so that each child receives at least one banana?

(a)	(b)
1) 1, 2, 3	1) $b \mid b \mid b \mid$
2) 1, 3, 3	2) $b \mid \mid b\ b \mid$
3) 3, 4, 4	3) $\mid \mid b \mid b\ b$
4) 4, 4, 4	4) $\mid \mid \mid b\ b\ b$

(a)	(b)
1) 1, 2, 2, 3, 3, 4	1) $o \mid o\ o \mid o\ o \mid o$
2) 1, 2, 2, 4, 4, 4	2) $o \mid o\ o \mid \mid o\ o\ o$
3) 2, 2, 2, 3, 3, 3	3) $\mid o\ o\ o \mid o\ o\ o \mid$
4) 4, 4, 4, 4, 4, 4	4) $\mid \mid \mid o\ o\ o\ o\ o\ o$

C(4 + 3 – 1, 3) × C(4 + 6 – 1, 6) = 20 × 84 = 1680

Examples

- **Example 1.34:** In how many ways can one distribute 10 (identical) white marbles among 6 distinct containers?

  C(6 + 10 – 1, 10) = 3003

- **Example 1.35:** How many such solutions are there to the inequality $x_1 + x_2 + \cdots + x_6 < 10$?

  We can transform the problem to

  $x_1 + x_2 + \cdots + x_6 + x_7 = 10,$

  $0 \le x_i, 1 \le i \le 6, 0 < x_7.$

  ➡ C(7 + 9 – 1, 9) = 5005

Binomial and Multinomial Expansions (1/2)

- **Example 1.36:** How many terms are there in the expansion of $(w + x + y + z)^{10}$.

  Each distinct term is of the form $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$, where $0 \le n_i$ for $1 \le i \le 4$, and $n_1 + n_2 + n_3 + n_4 = 10$.

  ➡ There are C(4 + 10 – 1, 10) = 286 terms.

Binomial and Multinomial Expansions (2/2)

- **Example 1.37:** Count the number of compositions for the number 7.

n = The Number of Summands in a Composition of 7		The Number of Compositions of 7 with n Summands	
(i)	$n = 1$	(i)	$\binom{6}{6}$
(ii)	$n = 2$	(ii)	$\binom{6}{5}$
(iii)	$n = 3$	(iii)	$\binom{6}{4}$
(iv)	$n = 4$	(iv)	$\binom{6}{3}$
(v)	$n = 5$	(v)	$\binom{6}{2}$
(vi)	$n = 6$	(vi)	$\binom{6}{1}$
(vii)	$n = 7$	(vii)	$\binom{6}{0}$

➡ For each positive integer m, there are $\sum_{k=0}^{m-1} \binom{m-1}{k} = 2^{m-1}$ compositions.

Computer Science Application

- **Example 1.39:** Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

  How many times is the **print** statement executed in this program segment?

$$\binom{20+3-1}{3} = \binom{22}{3} = 1540 \text{ times}$$

The Idea of a Run (1/2)

- **Example 1.41:** Find the total number of ways 5 E's and 10 O's can determine 7 runs. A run is a consecutive list of identical entries that are preceded and followed by different entries or no entries at all.

$$\underbrace{OO}_{\text{Run}}\ \underbrace{E}_{\text{Run}}\ \underbrace{OOOO}_{\text{Run}}\ \underbrace{EEE}_{\text{Run}}\ \underbrace{OOO}_{\text{Run}}\ \underbrace{E}_{\text{Run}}\ \underbrace{O}_{\text{Run}}$$

The Idea of a Run (2/2)

- **Example 1.41 (cont.):**

$$x_1 + x_3 + x_5 + x_7 = 5, \qquad x_1, x_3, x_5, x_7 > 0$$
$$x_2 + x_4 + x_6 = 10, \qquad x_2, x_4, x_6 > 0$$

$$w_1 + w_3 + w_5 + w_7 = 10, \qquad w_1, w_3, w_5, w_7 > 0$$
$$w_2 + w_4 + w_6 = 5, \qquad w_2, w_4, w_6 > 0$$

➡ C(4, 1)C(9, 7) + C(9, 6)C(4, 2)
= 144 + 504 = 648

Homework Assignment #1

- **EXERCISES 1.1 AND 1.2**
  11, 31, 34, 38
- **EXERCISES 1.3**
  20, 26
- **EXERCISES 1.4**
  18